

Eigen Vector and Eigen Value

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ABSTRACT

The word Eigen is understood in the German language with the meaning of own, proper, character. But in this discussion, instead of the above words, we use specific or Appropriation words. According to the above explanation, the following concepts are discussed. A definite vector, a definite value (a definite coefficient), a definite space, a definite equation, a definite polynomial, a definite matrix, and a definite function. The concept of Eigenvector and Eigenvalue has an important and special rule in linear algebra and functional analysis, which includes spectral theory.

Its eigenvectors are those vectors that are only stretched, with neither rotation nor shear. The corresponding eigenvalue is the factor by which an eigenvector is stretched or squished. If the eigenvalue is negative, the eigenvector's direction is reversed.

The eigenvectors and eigenvalues of a linear transformation serve to characterize it, and so they play important roles in all the areas where linear algebra is applied, from geology to quantum mechanics. In particular, it is often the case that a system is represented by a linear transformation whose outputs are fed as inputs to the same transformation (feedback). In such an application, the largest eigenvalue is of particular importance, because it governs the long-term behavior of the system after many applications of the linear transformation, and the associated eigenvector is the steady state of the system.

Keywords- Eigenvalues, Eigenvectors, Linear Transformation, Matrix Decomposition, Characteristic Equation.

I. INTRODUCTION

1.1 Definition:

The Eigenvector of square matrix A is the nonzero vector of x that satisfies the relation $Ax = \lambda x$ for scalar λ . Of course, the λ is called Eigenvalue of the square matrix A , if $Ax = \lambda x$ has a solution.

Example 1.1: Are the $u = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$ and $v = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ vectors

Eigenvector of the $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ square matrices?

Solve: Considering the equation $Au = \lambda u$ will have.

$$a; Au = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \cdot \begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -24 \\ -20 \end{pmatrix} = -4 \begin{pmatrix} 6 \\ -5 \end{pmatrix} = -4u$$

$$Av = \lambda v$$

$$b; Av = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \cdot \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -9 \\ 11 \end{pmatrix} \neq \lambda \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

So, we can say that the vector u is Eigenvector to the matrix whose Eigenvalue of $\lambda = -4$ is λ , but the vector v is not its Eigenvector.

1.2 Characteristic Spaces

A is an Eigenvalue if and only if the equation $(A - \lambda I)x = 0$ has a non-zero solution for x . So, the set of all solutions of the above equation, is the null space of the $A - \lambda I$ matrix, that's mean:

$$\lambda Nul(A - \lambda I) = \{x \in V: (A - \lambda I)x = 0\}.$$

The above space which depends on the value of λ , that's corresponding space called the characteristic space of A and is represented by E_λ that's mean: $E_\lambda = Nul(A - \lambda I)$.

Example 2.2: If 2 is the Eigenvalue of $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$

matrix, then find the basis of this matrix's characteristic space?

Solve: We have what.

$$Nul(A - \lambda I)x = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{vmatrix} - 2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \Rightarrow \begin{vmatrix} 2x_1 - x_2 + 6x_3 \\ 2x_1 - x_2 + 6x_3 \\ 2x_1 - x_2 + 6x_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}.$$

1.3 Characteristic Equation

An exponential scalar equation (Characteristic Equation) is very important for determining Eigenvalues. As an example, consider the following example.

Example 3.3: Determine the matrix $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ Eigenvalues?

Solve: We find all the λ scalars such that the equation $(A - \lambda I)x = 0$ has a non-zero solution.

Then.

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -\lambda-6 \end{bmatrix}.$$

So, the Eigenvalues of λ is the numerical solution of the above matrix, namely.

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 3 & -\lambda-6 \end{vmatrix} = \{(2-\lambda)(-\lambda-6) - 9\} \Rightarrow \lambda^2 + 4\lambda - 3 = 0$$

$$\lambda_1 = 1, \lambda_2 = 3.$$

That is, the Eigenvalues of matrix $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ are $\lambda_1 = 1, \lambda_2 = 3$.

So, we define the Characteristic Equation as; The quantity of λ is the Eigenvalue of the square matrix A , if and only if that the $\det(A - \lambda I) = 0$ will true for the equation λ , then the equation $\det(A - \lambda I) = 0$ is called a Characteristic Equation.

1.4 Characteristic polynomial

The function of p is called the Characteristic polynomial of the matrix A such that: $p(\lambda) = \det(A - \lambda I)$. We can, with the help of Characteristic polynomial and Characteristic Equation to determine Eigenvector and Eigenvalue.

We know that the solution condition of the equation $A - \lambda I = 0$ is $|A - \lambda I| = 0$ where $|A - \lambda I| = 0$ is the characteristic equation of A matrix and $p(\lambda) = |A - \lambda I|$ is called a Characteristic polynomial. In the above equations, the unknown number is λ and the polynomial form of the Characteristic equation is: $\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_n = 0$

And in the above polynomial the values of the unknowns are Eigenvalue.

Example 4.3: Find the Characteristic equation and Characteristic polynomial of the following matrix!

$$A = \begin{bmatrix} 5 & -2 & 6 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve: Considering the equation $\det A - \lambda I$, we have that.

$$\det \left(\begin{bmatrix} 5 & -2 & 6 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \Rightarrow \det \left(\begin{bmatrix} 5 & -2 & 6 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \right) \Rightarrow \det \left(\begin{bmatrix} 5-\lambda & -2 & 6 & 1 \\ 0 & 3-\lambda & -3 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} \right) = (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda) \Rightarrow p(\lambda) = \lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75.$$

As a result, we write that its Characteristic polynomial is $p(\lambda) = \lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75$ and its Characteristic equation is $(5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda) = 0$.

Now, if we solve characteristic polynomial or characteristic equation, then we will obtain Eigenvalues of the matrix.

1.5 Similarity metrics

If A and B are two $n \times n$ matrices, then A matrix is similar A matrix, if $P_{n \times n}$ matrix exists such that: $P^{-1}AP = B \sim A = PBP^{-1}$

And considering the similarity of A and B , the following relation holds true:

$$Q^{-1}QP = A \sim B = PAP^{-1}$$

That is, if A is similar B , then B is similar A and generally we say that these two matrices are similar to each other.

1.6 Diagonalization of a matrix

By applying Eigenvectors and Eigenvalues, we can decompose a matrix in a suitable way to the matrix $A = PDP^{-1}$. Such that the matrix D is a diagonal matrix in the above matrix.

Example 6.4: Consider the matrix $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$, Find a formula for A^k , such the matrixes $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$ are given and $A = PDP^{-1}$.

Solve; we have

$$P = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{\det P} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}. P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{-1} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

Then.

$$A = PDP^{-1} \Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$A^2 = (PDP^{-1})A^2$$

$$A^3 = (PDP^{-1})A^2$$

:

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

1.7 Quadratic forms

In this section, there is a discussion about those functions which are the intervals are square variables (the product of two variables) that's the Quadratic forms which are used for the solving of geometrical problems,

in vibration of mechanical systems and in electrical engineering.

We know that n unknowns' linear equation has the following form.

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b \quad (1)$$

The right-side expression of the above equation is:

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \quad (2)$$

Which is called a linear form as n variables function, such that all the variables in this equation have degree one and the product of these variables does not exist. The quadratic form of the variables x_1 and x_2 in the above expression is as follows:

$$a_1x_1^2 + a_2x_2^2 + a_3x_1x_2 \quad (3)$$

And the quadratic form of the variables x_1, x_2 and x_3 is as follows:

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_1x_2 + a_5x_1x_3 + a_6x_2x_3 \quad (4)$$

The boundaries of a quadratic form in which obtained the product of two variables is called the cross product term. The last term of the third expression and the last term of the fourth expression are the cross product terms.

We can also express the third and fourth quadratic expressions in the following matrix form:

$$(x_1 \ x_2) \begin{bmatrix} a_1 & a_3/2 \\ a_3/2 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (5)$$

$$(x_1 \ x_2 \ x_3) \begin{bmatrix} a_1 & a_4/2 & a_5/2 \\ a_4/2 & a_2 & a_6/2 \\ a_5/2 & a_6/2 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (6)$$

The product of matrixes (5) and (5) have the product case of X^TAX , where X is a matrix and A is a symmetric matrix, whose diagonal elements are the coefficients of the square terms, and the symmetric elements are the coefficients of the cross terms.

1.8 Definition of quadratic form

A quadratic form in the space $\mathbb{R}^n$ is a function of Q as defined in the space $\mathbb{R}^n$. And for x vector in $\mathbb{R}^n$ space, $Q(x) = X^TAX$ has value.

Thus, the A symmetric matrix is also called the matrix of the quadratic form. And in special cases, the Q function chooses the following form for the identity matrix:

$$Q(x) = X^TIX = (x, x) = |x|^2$$

Example 7.5: In matrix form, the four quadratic forms are given below:

$$2x^2 + 2xy - 7y^2 = (x \ y) \begin{pmatrix} 2 & 3 \\ 3 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (1)$$

$$4x^2 - 5y^2 = (x \ y) \begin{pmatrix} 4 & 0 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (2)$$

$$xy = (x \ y) \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (3)$$

$$x_1^2 + 7x_2^2 - 3x_3^2 + 4x_1x_2 - 2x_1x_3 + 6x_2x_3 =$$

$$(x_1 \ x_2 \ x_3) \begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & 3 \\ -1 & 3 & -3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (4)$$

Note: In quadratic systems, symmetrical matrices and their applications are useful. The matrix representation of a quadratic form is unique such that the corresponding matrix is symmetric. The first quadratic form of the above example is shown below.

$$2x^2 + 6xy - 7y^2 = (x \ y) \begin{pmatrix} 2 & 5 \\ 1 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

1.9 The transform of variables into a quadratic form

When x is a vector of $\mathbb{R}^n$ then the transformation equation of the variable is as follows:

$$x = Py \sim y = P^{-1}x \quad (6)$$

such that P is an inverse matrix and y is a vector of $\mathbb{R}^n$.

Considering the transformation of the variable X^TAX the quadratic form takes the following form

$$X^TAX = (Py)^T A(Py) = y^T P^T A Py = y^T (P^T A P) y \quad (7)$$

The new matrix of the quadratic form is P^TAP , of course, if P diagonalizes the A matrix vertically, that it is diagonal and orthogonal, then $P^T = P^{-1}$ and.

$$P^TAP = P^{-1}AP = D \quad (8)$$

Therefore, the matrix of the given quadratic form is diagonalized, by using a diagonal matrix deleting the cross-product term from the quadratic expression which is the main purpose of our discussion.

Example 8.6: Consider the following quadratic form and find a transformation of a variable such that the cross-product term $(-8x_1x_1)$ is deleted.

$$Q(x) = x_1^2 - 8x_1x_1 - 5x_2^2.$$

Solve; It is obvious that.

$$Q(x) = x_1^2 - 8x_1x_1 - 5x_2^2 = (x_1 \ x_2) \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow$$

$$A = \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix}$$

Now let's diagonalize the A matrix.

First, we find the eigenvalues of $A = \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix}$ which are $\lambda_1 = 3$ and $\lambda_2 = -7$, so that the eigenvectors of this matrix are:

$$\lambda_1 = 3: v_1 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}, \lambda_2 = -7: v_2 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

The above vectors are orthogonal because the distinct eigenvectors of the A matrix are as follows:

$$P = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}, D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$$

Then.

$$A = PDP^{-1} \Rightarrow D = P^{-1}AP = P^TAP: x = Py, x =$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Accordingly.

$$x_1^2 - 8x_1x_1 - 5x_2^2 = x^T Ax = (Py)^T A(Py) = y^T P^T A Py = y^T (P^T A P) y = y^T D y = 3y_1^2 - 8y_2^2$$

Now let's define the y vector:

$$x = Py \Rightarrow y = P^{-1}x = P^T x = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

II. METHODOLOGY

Since the collection of materials and information in this article is in the form of a library, therefore a qualitative method has been used. and have been used in the collection of materials and information reliable, national and international sources and references. I have also benefited from the advice and guidance of experienced lecturers in compiling these materials.

In this research paper, we studied Eigen Vector, Eigen Value and their application and importance in linear algebra.

III. CONCLUSION

The concepts of eigenvalues and eigenvectors play a central role in understanding the structure and behavior of linear transformations. These quantities reveal intrinsic properties of matrices that remain unchanged under transformation, and they provide deep insight into systems modeled by linear operators. Through diagonalization, the complexity of matrix computations can be reduced, simplifying the analysis of stability, vibration, and dynamic behavior in various engineering and physical systems. Moreover, the study of quadratic forms and orthogonal transformations demonstrates how eigenvalues determine the geometric and analytic features of functions and matrices. Overall, Eigen theory serves as a bridge between abstract mathematics and practical applications across physics, engineering, computer science, and data analysis.

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