

Extensions of Hermite - Hadamard Inequalities for Semi-P-Functions

Hedayatullah Baheer¹, Abdul Wakil Baidar² and Noorullah Noori³

¹Department of Higher Mathematics, Faculty of Electromechanic, Kabul Polytechnic University, Kabul, AFGHANISTAN

²Department of Mathematics, Faculty of Mathematical Science, Kabul University, Kabul, AFGHANISTAN

³Department of Mathematics, Faculty of Mathematical Science, Kabul University, Kabul, AFGHANISTAN

¹Corresponding Author: hedayat.baheer2012@gmail.com

ORCID

<https://orcid.org/0009-0008-8279-7979>



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ABSTRACT

In this paper, several extended Hermite–Hadamard type inequalities are established for semi P-functions. By employing the extended Hermite–Hadamard inequality, new integral estimates are derived under the semi P-convexity framework. In addition, Hölder’s inequality and the power mean inequality are applied to obtain further bounds and refinements for the corresponding integral expressions. Some of the obtained results provide extensions and generalizations of existing Hermite–Hadamard type inequalities for semi P-functions. Furthermore, applications of the obtained inequalities to several classical means are presented, illustrating their usefulness in establishing relationships and bounds among different means. The results demonstrate that the semi P-function framework, together with Hölder’s inequality and the power mean inequality, provides a flexible approach for deriving refined integral inequalities and their applications.

Keywords- Hermite–Hadamard inequality; semi-(P)-functions; extended Hermite–Hadamard inequality; Hölder’s inequality; power mean inequality; integral inequalities; means.

I. INTRODUCTION

The Hermite–Hadamard inequality, discovered independently by Hermite and Hadamard [1], is one of the most fundamental results in convex analysis. If $f: [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ is a convex function,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2} \quad (1.1)$$

This double inequality gives two-sided estimates for the integral mean of a convex function and has many applications in numerical integration, optimization, probability theory, and the theory of means. Because of its importance, numerous extensions and refinements have been obtained by replacing convexity with weaker or related notions. Among these, P-functions, ss-convex

functions, h-convex functions, mm-convex functions, (α, m) -convex functions, multiplicatively P-functions, (s, P) -functions, and exponential type P-functions have attracted considerable attention; see [2–15] and the references therein.

A particularly useful generalization is the class of P-functions. A nonnegative function $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ is called a P-function if

$$f(tx + (1-t)y) \leq f(x) + f(y) \quad (1.2)$$

for all $x, y \in I$ and $t \in (0, 1)$. For such functions,

Dragomir, Pečarić and Persson [2] proved the Hermite–Hadamard type inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_a^b f(x) dx \leq 2[f(a) + f(b)] \quad (1.3)$$

This result opened the way to many related inequalities for P-type functions.

Recently, Kadakal, İşcan and Kadakal [3] introduced the concept of semi-P-functions. A function $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ is called a semi-P-function if

$$f(tx + (1-t)y) \leq (tx + (1-t)y)[f(x) + f(y)] \quad (1.4)$$

for all $x, y \in I$ and any $t \in [0,1]$. This definition modifies the usual P-function inequality by inserting the convex combination $tx + (1-t)y$ as a weight. Consequently, the position of the intermediate point influences the upper bound, which makes semi-P-functions more flexible than ordinary P-functions. Under suitable domain restrictions, many important classes of functions are semi-P-functions, including nonnegative convex functions on $[1, \infty)$, P-functions on $[1, \infty)$, quasi-convex functions, ss-convex functions in the first and second senses, trigonometrically convex functions, log-convex functions, and arithmetic-harmonic convex functions; see [3].

For semi-P-functions, the following Hermite-Hadamard type inequality was established in [3]:

$$\begin{aligned} \frac{1}{a+b} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f(x) dx \\ &\leq (a+b) \frac{f(a) + f(b)}{2} \quad (1.5) \end{aligned}$$

This result generalizes the corresponding inequality for P-functions and provides a new framework for integral estimates. However, this basic inequality only bounds the integral mean. For differentiable functions, much sharper results can be obtained by using derivative-based identities and classical integral inequalities.

A fundamental tool in this direction is the Dragomir-Agarwal identity [4]:

$$\begin{aligned} \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \\ = \frac{b-a}{2} \int_0^1 (1-2t) f'(ta \\ + (1-t)b) dt \quad (1.6) \end{aligned}$$

By combining this identity with Hölder's inequality, the power-mean inequality, the improved power-mean inequality, and the Hölder-İşcan inequality, many authors have derived Hermite-Hadamard type estimates for functions whose derivatives in absolute value belong to various convexity classes. For example, Xi and Qi [5] established general Hermite-Hadamard type inequalities for differentiable convex functions via a parameterized identity, which contains trapezoidal, midpoint, and Simpson-type inequalities as special cases. Kirmaci [6] obtained midpoint and trapezoidal estimates for differentiable mappings. Pearce and Pečarić [9] improved some constants in earlier inequalities. Dragomir and Fitzpatrick [10] proved a variant of Hadamard's inequality for s-convex functions in the second sense. More recently, Numan and İşcan studied (s,P)-functions

[11] and exponential type P-functions [12]. Ozcan [13] investigated multiplicatively h-convex functions, while Bai, Qi and Xi [14] considered m- and (α, m) -logarithmically convex functions. Sarikaya, Set and Ozdemir [15] obtained Simpson-type inequalities for ss-convex functions. These works show that the choice of integral inequality and the convexity class strongly affects the sharpness of the resulting bounds.

Despite these developments, Hermite-Hadamard inequalities for semi-P-functions in the differentiable setting have not yet been fully explored. The main purpose of this paper is to extend Hermite-Hadamard inequalities for semi-P-functions. More precisely, we establish new estimates for differentiable functions whose first derivative in absolute value, or whose power, is a semi-P-function. Using Hölder, power-mean, improved power-mean, and Hölder-İşcan integral inequalities, we obtain bounds that refine and extend the basic Hermite-Hadamard inequality for semi-P-functions. We also compare the obtained results and show that the Hölder-İşcan and improved power-mean approaches provide better estimates than some classical ones. Finally, we apply the main results to special means of real numbers, including arithmetic, geometric, harmonic, logarithmic, p-logarithmic, and identric means. Our results generalize and complement several known inequalities for P-functions, (s,P)-functions, exponential type P-functions, and related classes.

II. METHODOLOGY

2.1 Research Design

This study follows a theoretical and analytical approach. The main objective is to extend Hermite-Hadamard inequalities for semi-P-functions by deriving new integral estimates for differentiable mappings whose first derivative, in absolute value or in a certain power, is a semi-P-function. The methodology is based on classical integral identities, convexity-type assumptions, and well-known integral inequalities such as Hölder, power-mean, improved power-mean, and Hölder-İşcan inequalities. No numerical experiments are performed; all results are established by rigorous mathematical proofs.

Preliminaries and Fundamentals

The function $f: [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$, is said to be convex if the following inequality holds

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad (2.1)$$

for all $x, y \in [a, b]$ and $t \in [0,1]$. We say that f is concave if $(-f)$ is convex.

Let $I \subseteq \mathbb{R}$ be an interval and let $a, b \in I$ with $a < b$. A function $f: I \rightarrow \mathbb{R}$ is called a semi-P-function if

$$f(tx + (1-t)y) \leq (tx + (1-t)y)[f(x) + f(y)] \quad (2.2)$$

for all $x, y \in I$ and $t \in [0,1]$. We denote the class of semi-P-functions on I by $SP(I)$. Throughout the paper, we assume that $f \in L[a, b]$ whenever integrals are considered. In the differentiable case, we assume that f is differentiable on I° and $f' \in L[a, b]$.

Lemma2.1[14]: Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I^\circ$ with $a < b$

$$\begin{aligned} & f' \in L[a, b] \text{ and } \lambda, \mu \in \mathbb{R}, \text{ then} \\ & \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) \\ & - \frac{1}{b-a} \int_a^b f(x) dx \\ & = \frac{b-a}{4} \int_0^1 \left[(1-\lambda-t)f'(ta) \right. \\ & \quad \left. + (1-t)f'\left(t\frac{a+b}{2}\right) \right. \\ & \quad \left. + (\mu-t)f'\left(t\frac{a+b}{2} + (1-t)b\right) \right] dt \quad (2.3) \end{aligned}$$

Lemma2.2. [14] : For $s > 0$ and $0 \leq \xi \leq 1$, one has

$$\int_0^1 |\xi - t|^s dt = \frac{\xi^{s+1} + (1-\xi)^{s+1}}{s+1} \quad (2.4)$$

$$\int_0^1 t|\xi - t|^s dt = \frac{\xi^{s+2} + (s+1+\xi)(1-\xi)^{s+1}}{(s+1)(s+2)} \quad (2.5)$$

2.3 Basic Hermite–Hadamard Inequalities for Semi-P-Functions

We first establish the basic Hermite–Hadamard type inequality for semi-P-functions. Integrating the definition of a semi-P-function with respect to $t \in [0, 1]$ and using the change of variable $x = ta + (1-t)b$, we obtain

$$\begin{aligned} \frac{1}{a+b} f\left(\frac{a+b}{2}\right) & \leq \frac{1}{b-a} \int_a^b f(x) dx \\ & \leq (a+b) \frac{f(a) + f(b)}{2} \quad (2.6) \end{aligned}$$

This double inequality provides the foundation for the subsequent refinements.

2.4 Differentiable Case and Integral Identity

To obtain sharper estimates, we employ the Dragomir–Agarwal identity [4]:

$$\begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \\ & = \frac{b-a}{2} \int_0^1 (1-2t) f'(ta) \\ & \quad + (1-t)b) dt \quad (2.7) \end{aligned}$$

This identity expresses the difference between the trapezoidal rule and the integral mean in terms of the derivative f' . By assuming that $|f'|$ or $|f'|^q$ is a semi-P-function, we can estimate the right-hand side using integral inequalities.

2.5 Estimation via Integral Inequalities

We apply four main integral inequalities:

Hölder inequality [7]:

$$\begin{aligned} & \int_a^b |u(x)v(x)| dx \\ & \leq \left(\int_a^b |u(x)|^p dx \right)^{\frac{1}{p}} \left(\int_a^b |v(x)|^q dx \right)^{\frac{1}{q}} \quad (2.8) \end{aligned}$$

Power-mean inequality [8]:

$$\begin{aligned} & \int_a^b |u(x)v(x)| dx \\ & \leq \left(\int_a^b |u(x)|^p dx \right)^{1-\frac{1}{q}} \left(\int_a^b |u(x)||v(x)|^q dx \right)^{\frac{1}{q}} \quad (2.9) \end{aligned}$$

1. Improved power-mean inequality [8]: A refinement of the power-mean inequality that provides better estimates by splitting the integral over $[a, b]$ into two parts.

2. Hölder–İşcan inequality [7]: A refinement of Hölder's inequality that often yields sharper bounds than the classical Hölder inequality.

For each inequality, we substitute the semi-P property of $|f'|$ or $|f'|^q$, namely

$$\begin{aligned} |f(ta + (1-t)b)| & \leq (ta + (1-t)b)[|f'(a)| \\ & \quad + |f'(b)|] \quad (2.10) \end{aligned}$$

$$\begin{aligned} \text{Or} \\ |f(ta + (1-t)b)|^q & \leq (ta + (1-t)b)[|f'(a)|^q \\ & \quad + |f'(b)|^q] \quad (2.11) \end{aligned}$$

Then we evaluate the resulting integrals explicitly, such as

$$\int_0^1 |1-2t|(ta + (1-t)b) dt = \frac{a+b}{4} \quad (2.12)$$

and obtain new upper bounds for the Hermite–Hadamard difference.

2.6 Comparison and Sharpness

After deriving the new inequalities, we compare them with existing results. In particular, we show that the bounds obtained via the improved power-mean inequality and the Hölder–İşcan inequality are sharper than those obtained from the classical Hölder and power-mean inequalities. The comparison is carried out by using the concavity of the function $h(x) = x^\lambda$, $0 < \lambda \leq 1$, on $[0, \infty)$, and proving that the new bound is less than or equal to the old one.

2.7 Applications to Special Means

Finally, we apply the main inequalities to special means of real numbers. For two positive numbers a, b with $a < b$, we consider:

1. The arithmetic mean

$$A := A(a, b) = \frac{a+b}{2}, a, b \geq 0$$

2. The geometric mean

$$G := G(a, b) = \sqrt{ab}, a, b \geq 0$$

3. The harmonic mean

$$H := H(a, b) = \frac{2ab}{a+b}, a, b > 0$$

4. The logarithmic mean

$$L := L(a, b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, a \neq b \\ a, a = b \end{cases} a, b > 0$$

5. The P-logarithmic mean

$$L_p := L_p(a, b) = \begin{cases} \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}}, a \neq b, p \in \mathbb{R} \setminus \{-1, 0\} \\ a, a = b \end{cases} a, b > 0$$

6. The identric mean

$$I := I(a, b) = \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}}, a, b > 0$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature: $H \leq G \leq L \leq I \leq A$. It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Lemma 2.3: Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I^\circ$ with $a < b$ and assume that $f' \in L[a, b]$. If $|f'|$ is a semi P -function on interval $[a, b]$, then the following inequality holds

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} A(a, b) A(|f'(a)|, |f'(b)|) \quad (2.13)$$

In this inequality for shortness, we denote by $A(a, b)$ the arithmetic mean of real numbers a and b .

Theorem 2.1: Let $f: I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , and let $a, b \in I^\circ$ satisfy $a < b$. Suppose that the function $|f'(x)|$ is convex on the interval $[a, b]$. Then the following inequality is satisfied:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|) \quad (2.14)$$

Lemma 2.4: let $f, g: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$. If f and g are semi- P -functions, then $f + g$ is a semi P -function, cf is a semi p -functions. see [3]

Lemma 2.5: If $f: [1, \infty) \rightarrow [0, \infty)$ is a convex function, then f is also semi P -function. Since $t \leq ta + (1-t)b$, $1-t \leq ta + (1-t)b$, for all $a, b \in [1, \infty)$ and $t \in [0, 1]$, we get

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \leq (ta + (1-t)b)[f(a) + f(b)] \quad (2.15)$$

Lemma 2.6: If $f: [1, \infty) \rightarrow [0, \infty)$ is a P -function, then f is also a semi P -function. Since,

$$1 \leq ta + (1-t)b, \text{ for all } a, b \in [1, \infty) \text{ and } t \in [0, 1], \text{ we obtain}$$

$$f(ta + (1-t)b) \leq f(a) + f(b) \leq (ta + (1-t)b)[f(a) + f(b)] \quad (2.16)$$

See [3]

Lemma 2.7: Let $f: [1, \infty) \rightarrow [0, \infty)$ be a nonnegative and $s \in (0, 1]$. If f is a s -convex function in the first sense, then f is also a semi P -function. Since,

$$ts \leq 1 \leq ta + (1-t)b \text{ and } 1-ts \leq 1 \leq ta + (1-t)b,$$

$$\text{for all } a, b \in [1, \infty) \text{ and } t \in [0, 1], \text{ we get} \\ f(ta + (1-t)b) \leq t^s f(a) + (1-t^s)f(b) = (ta + (1-t)b)[f(a) + f(b)] \quad (2.17)$$

See [3]

Lemma 2.8: Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a semi P -function. If $a < b$ and f is Lebesgue integrable on $[a, b]$, then f satisfies the following Hermite-Hadamard type inequalities:

$$\frac{1}{a+b} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq (a+b) \frac{f(a) + f(b)}{2} \quad (2.18)$$

For a more comprehensive discussion and additional results concerning these functions, see [3]

Lemma 2.9[14]: Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I^\circ$ with $a < b$, $\lambda \geq 0, \mu \leq 1$ and $f' \in L[a, b]$. If $|f'(x)|^q$ for $q \geq 1$ is convex on $[a, b]$, then

$$\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} \left(\frac{1}{6}\right)^{\frac{1}{q}} \left\{ (1-2\lambda+2\lambda^2)^{1-\frac{1}{q}} \times [(4-9\lambda+12\lambda^2-2\lambda^3)|f'(a)|^q] \right. \\ \left. + (2-3\lambda+2\lambda^3)|f'(b)|^q \right]^{\frac{1}{q}} + (1-2\mu+2\mu^2)^{1-\frac{1}{q}} \times [(2-3\mu+2\lambda^3)|f'(a)|^q] \\ \left. + (4-9\mu+12\mu^2-2\mu^3)|f'(b)|^q \right]^{\frac{1}{q}} \quad (2.19)$$

Lemma2.10[14]: Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'(x)|$ is convex on $[a, b]$, then

$$\left| \frac{1}{2} \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{16} (|f'(a)| + |f'(b)|) \quad (2.20)$$

$$\left| \frac{1}{3} \left[f(a) + f(b) + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{5(b-a)}{144} (|f'(a)| + |f'(b)|) \quad (2.21)$$

$$\left| \frac{1}{6} \left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{5(b-a)}{72} (|f'(a)| + |f'(b)|) \quad (2.22)$$

Lemma2.11[14]: Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'(x)|$ is convex on $[a, b]$, then

$$\left| \frac{1}{2} \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{16} (|f'(a)| + |f'(b)|) \quad (2.23)$$

$$\left| \frac{1}{6} \left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{2(b-a)}{27} (|f'(a)| + |f'(b)|) \quad (2.24)$$

Lemma2.12[14]: Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'(x)|^q$ is convex for $q \geq 1$ on $[a, b]$ and

$$\frac{f(a) + f(b)}{2} = f\left(\frac{a+b}{2}\right)$$

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| = \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{16} \left(\frac{1}{4(q+1)(q+2)} \right)^{\frac{1}{q}} \left\{ [2q + 5]|f'(a)|^q + (2q+3)|f'(b)|^q \right\}^{\frac{1}{q}} + [|f'(a)|^q + (4q+7)|f'(b)|^q]^{\frac{1}{q}} + [(4q+7)|f'(a)|^q + |f'(b)|^q]^{\frac{1}{q}} + [(2q+3)|f'(a)|^q + (2q+5)|f'(b)|^q]^{\frac{1}{q}} \quad (2.25)$$

Lemma2.13: Let $f: I \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I$ with $a < b$, and assume that $f' \in L[a, b]$. If

$|f'|$ is a semi P-function on interval $[a, b]$, then the following inequality holds

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} A^{\frac{1}{q}}(a, b) A^{\frac{1}{q}}(|f'(a)|^q, |f'(b)|^q) \quad (2.26)$$

See[3]

III. PRIOR APPROACH

The Hermite–Hadamard inequality, established independently by Hermite and Hadamard in the late nineteenth century [1], is one of the most classical and widely used results in convex analysis. For a convex function $f: [a, b] \rightarrow \mathbb{R}$, it states that

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2} \quad (3.1)$$

This double inequality provides a two-sided estimate for the integral mean of a convex function and has motivated a vast literature on generalizations, refinements, and applications. Many researchers have extended it by replacing convexity with weaker or related notions, such as P-functions, s-convex functions, h-convex functions, m-convex functions, (α, m) -convex functions, multiplicatively P-functions, and exponential type P-functions. Each of these classes leads to new integral inequalities and often yields sharper estimates for differentiable mappings.

A particularly important generalization is the class of P-functions. A nonnegative function $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called a P-function if

$$f(tx + (1-t)y) \leq f(x) + f(y) \quad (3.2)$$

for all $x, y \in I$ and $t \in (0, 1)$. Dragomir, Pečarić and Persson [2] established the following Hermite–Hadamard type inequality for P-functions:

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_a^b f(x) dx \leq 2[f(a) + f(b)] \quad (3.3)$$

This result paved the way for many related inequalities. However, the constant 2 on the right-hand side is relatively large, and the inequality does not capture the influence of the position of the intermediate point $tx + (1-t)y$. This limitation motivated the introduction of more refined classes of functions.

In recent years, several authors have studied s-convex functions in the first and second senses, h-convex functions, and m-convex functions. For example, Dragomir and Fitzpatrick [8] proved a variant of Hadamard's inequality for ss-convex functions in the second sense:

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{s+1} \quad (3.4)$$

Kirmaci [6] obtained midpoint and trapezoidal estimates for differentiable mappings, while Pearce and Pečarić [7] improved some constants in earlier inequalities. Xi and Qi [5] established general Hermite–Hadamard type inequalities for differentiable convex functions via a parameterized identity, which includes trapezoidal, midpoint, and Simpson-type inequalities as special cases. Sarikaya, Set and Ozdemir [12] derived Simpson-type inequalities for ss -convex functions. Numan and İşcan [9] studied (s,P) -functions and later exponential type P -functions [10]. Ozcan [11] investigated multiplicatively h -convex functions, and Bai, Qi and Xi [13] considered m - and (α,m) -logarithmically convex functions. These works show that the choice of convexity class and the integral inequality used strongly affect the sharpness of the resulting bounds.

Despite these developments, a common drawback of many existing approaches is that the upper bound in the Hermite–Hadamard inequality does not depend on the location of the point $tx + (1-t)y$. This observation led Kadakal, İşcan and Kadakal [3] to introduce the concept of semi- P -functions. A function $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called a semi- P -function if

$$f(tx + (1-t)y) \leq (tx + (1-t)y)[f(x) + f(y)] \quad (3.5)$$

for all $x, y \in I$ and $t \in [0,1]$. This definition modifies the usual P -function inequality by inserting the convex combination $tx + (1-t)y$ as a weight. Consequently, the position of the intermediate point influences the upper bound, which makes semi- P -functions more flexible than ordinary P -functions. Under suitable domain restrictions, many important classes of functions are semi- P -functions, including nonnegative convex functions on $[1, \infty)$, P -functions on $[1, \infty)$, quasi-convex functions, ss -convex functions in the first and second senses, trigonometrically convex functions, log-convex functions, and arithmetic–harmonic convex functions [3].

For semi- P -functions, the following basic Hermite–Hadamard type inequality was established in [3]:

$$\begin{aligned} \frac{1}{a+b} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f(x) dx \\ &\leq (a+b) \frac{f(a)+f(b)}{2} \quad (3.6) \end{aligned}$$

This result generalizes the corresponding inequality for P -functions and provides a new framework for integral estimates. However, this basic inequality only bounds the integral mean; it does not exploit differentiability of the function. For differentiable

functions, much sharper results can be obtained by using derivative-based identities and classical integral inequalities.

A fundamental tool in this direction is the Dragomir–Agarwal identity [4]:

$$\begin{aligned} \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \\ = \frac{b-a}{2} \int_0^1 (1-2t) f'(ta \\ + (1-t)b) dt \quad (3.7) \end{aligned}$$

By combining this identity with Hölder’s inequality, the power-mean inequality, the improved power-mean inequality, and the Hölder–İşcan inequality, many authors have derived Hermite–Hadamard type estimates for functions whose derivatives in absolute value belong to various convexity classes. For instance, Xi and Qi [5] used a parameterized identity to obtain general inequalities, while Kirmaci [6] and Pearce and Pečarić [7] focused on midpoint and trapezoidal rules. These techniques, however, have not yet been fully developed for semi- P -functions. In particular, no derivative-based refinements of the Hermite–Hadamard inequality for semi- P -functions have been established so far.

The present paper aims to fill this gap. We extend Hermite–Hadamard inequalities for semi- P -functions by deriving new estimates for differentiable functions whose first derivative in absolute value, or whose power, is a semi- P -function. Using Hölder, power-mean, improved power-mean, and Hölder–İşcan integral inequalities, we obtain bounds that refine and extend the basic Hermite–Hadamard inequality for semi- P -functions. We also compare the obtained results and show that the Hölder–İşcan and improved power-mean approaches provide better estimates than some classical ones. Finally, we apply the main results to special means of real numbers. Our results generalize and complement several known inequalities for P -functions, (s,P) -functions, exponential type P -functions, and related classes.

IV. OUR APPROACH

Theorem 4.1. Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I^\circ$ with $a < b$ and assume that $f' \in L[a, b]$. If $|f'|$ is a semi P -function on interval $[a, b]$, then the following inequality holds

$$\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{24} A(|f'(a)|, |f'(b)|) \{ (4 - 9\lambda + 12\lambda^2 - 2\lambda^3)a + (2 - 3\lambda + 2\lambda^3)b + (2 - 3\mu + 2\mu^3)a + (4 - 9\mu + 12\mu^2 - 2\mu^3)b \} \quad (4.1)$$

Proof: using Lemma 2.1 and the inequality

$$\begin{aligned} \left| f'\left(ta + (1-t)\frac{a+b}{2}\right) \right| &= \left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right| \\ &\leq \left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) [|f'(a)| + |f'(b)|] \\ \left| f'\left(t\frac{a+b}{2} + (1-t)b\right) \right| &= \left| f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right| \\ &\leq \left(\frac{t}{2}a + \frac{2-t}{2}b\right) [|f'(a)| + |f'(b)|] \end{aligned}$$

We get

$$\begin{aligned} &\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ &= \frac{b-a}{4} \left[\int_0^1 |1 - \lambda - t| \left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) [|f'(a)| + |f'(b)|] dt \right. \\ &\quad \left. + \int_0^1 |\mu - t| \left(\frac{t}{2}a + \frac{2-t}{2}b\right) [|f'(a)| + |f'(b)|] dt \right] \\ &\leq \frac{b-a}{4} \left[\int_0^1 |1 - \lambda - t| \left| f'\left(ta + (1-t)\frac{a+b}{2}\right) \right| dt \right. \\ &\quad \left. + \int_0^1 |\mu| \right. \\ &\quad \left. - t \left| f'\left(t\frac{a+b}{2} + (1-t)b\right) \right| dt \right] \end{aligned}$$

$$= \frac{b-a}{4} [|f'(a)| + |f'(b)|] \left\{ \int_0^1 |1 - \lambda - t| \left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt + \int_0^1 |\mu - t| \left(\frac{t}{2}a + \frac{2-t}{2}b\right) dt \right\} \quad (4.2)$$

In virtue of Lemma 2.2, a direct calculation. Yields

$$\begin{aligned} &\int_0^1 |1 - \lambda - t| \left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt \\ &= \int_0^1 |1 - \lambda - t| \left(\frac{1}{2}(a+b) + \frac{1}{2}(a-b)t\right) dt \\ &= \frac{1}{2}(a+b) \int_0^1 |1 - \lambda - t| dt + \frac{1}{2}(a-b) \int_0^1 t |1 - \lambda - t| dt \\ &= \frac{1}{2}(a+b) \left(\frac{1}{2} - \lambda + \lambda^2\right) + \frac{1}{12}(a-b)[(1-\lambda)^3 + (1-\lambda)^2(3-\lambda)] \\ &= \frac{1}{12}(4 - 9\lambda + 12\lambda^2 - 2\lambda^3)a + \frac{1}{12}(2 - 3\lambda + 2\lambda^3)b \\ &\quad + \int_0^1 |\mu - t| \left(\frac{t}{2}a + \frac{2-t}{2}b\right) dt \\ &= \int_0^1 |\mu - t| \left(\frac{1}{2}(a-b)t + b\right) dt \\ &= \frac{1}{2}(a-b) \int_0^1 t |\mu - t| dt + b \int_0^1 |\mu - t| dt \\ &= \frac{1}{12}(a-b)[\mu^3 + (1-\mu)^2(2+\mu)] + b\left(\frac{1}{2} - \mu + \mu^2\right) \\ &= \frac{1}{12}(2 - 3\mu + 2\mu^3)a + \frac{1}{12}(4 - 9\mu + 12\mu^2 - 2\mu^3)b \end{aligned}$$

Substituting the above two equalities into inequality 4.2 and utilizing the result of Lemma 2.2 yields inequality 1 for $q > 1$.

$$\begin{aligned} & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) \right. \\ & \quad \left. - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4} [|f'(a)| \\ & \quad + |f'(b)|] \left\{ \frac{1}{12} (4 - 9\lambda + 12\lambda^2 \right. \\ & \quad - 2\lambda^3)a + \frac{1}{12} (2 - 3\lambda + 2\lambda^3)b \\ & \quad + \frac{1}{12} (2 - 3\mu + 2\mu^3)a \\ & \quad \left. + \frac{1}{12} (4 - 9\mu + 12\mu^2 - 2\mu^3)b \right\} \\ & = \frac{b-a}{24} A(|f'(a)|, |f'(b)|) \{ (4 - 9\lambda + 12\lambda^2 - 2\lambda^3)a \\ & \quad + (2 - 3\lambda + 2\lambda^3)b \\ & \quad + (2 - 3\mu + 2\mu^3)a \\ & \quad + (4 - 9\mu + 12\mu^2 - 2\mu^3)b \} \end{aligned}$$

If $\lambda = \mu = 1$ we can get lemma 2.3 exactly.

Theorem 4.2 Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I^\circ$ with $a < b, \lambda \geq 0, \mu \leq 1, q > 1, \frac{1}{p} + \frac{1}{q} = 1$, and assume that $f' \in L[a, b]$. If $|f'|^q$ is a semi P-function on interval $[a, b]$, then the following inequality holds

$$\begin{aligned} & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) \right. \\ & \quad \left. - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2} A^{\frac{1}{q}}(|f'(a)|^q, |f'(b)|^q) \\ & \quad \left\{ \left(\frac{(1-\lambda)^{p+1} + \lambda^{p+1}}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{3a+b}{4} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{\mu^{p+1} + (1-\mu)^{p+1}}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{a+3b}{4} \right)^{\frac{1}{q}} \right\} \quad (4.3) \end{aligned}$$

Proof. Using Lemma 2.1, Holder's integral inequality and the following inequality

$$\begin{aligned} & \left| f'\left(ta + (1-t)\frac{a+b}{2}\right) \right|^q = \left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right|^q \\ & = \left(\frac{1+t}{2}a + \frac{1-t}{2}b \right) [|f'(a)|^q \\ & \quad + |f'(b)|^q] \end{aligned}$$

and

$$\begin{aligned} & \left| f'\left(t\frac{a+b}{2} + (1-t)b\right) \right|^q = \left| f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right|^q \\ & \leq \left(\frac{t}{2}a + \frac{2-t}{2}b \right) [|f'(b)|^q \\ & \quad + |f'(a)|^q] \end{aligned}$$

Which is the semi P-function of $|f'|^q$, we have

$$\begin{aligned} & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) \right. \\ & \quad \left. - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4} \left[\int_0^1 |1 - \lambda - t| \left| f'\left(ta + (1-t)\frac{a+b}{2}\right) \right| \right. \\ & \quad \left. + \int_0^1 |\mu \right. \\ & \quad \left. - t| \left| f'\left(t\frac{a+b}{2} + (1-t)b\right) \right| dt \right] \\ & \leq \frac{b-a}{4} \left[\left(\int_0^1 |1 - \lambda \right. \right. \\ & \quad \left. \left. - t|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'\left(ta \right. \right. \right. \\ & \quad \left. \left. + (1-t)\frac{a+b}{2} \right|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^1 |\mu \right. \right. \\ & \quad \left. \left. - t|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'\left(t\frac{a+b}{2} \right. \right. \right. \\ & \quad \left. \left. + (1-t)b \right|^q dt \right)^{\frac{1}{q}} \right] \\ & \leq \frac{b-a}{4} \left[\left(\int_0^1 |1 - \lambda - t|^p dt \right)^{\frac{1}{p}} \left([|f'(a)|^q \right. \right. \\ & \quad \left. \left. + |f'(b)|^q] \int_0^1 \left(\frac{1+t}{2}a \right. \right. \right. \\ & \quad \left. \left. + \frac{1-t}{2}b \right) dt \right)^{\frac{1}{q}} \\ & \quad \left. + \left(\int_0^1 |\mu - t|^p dt \right)^{\frac{1}{p}} \left([|f'(a)|^q \right. \right. \\ & \quad \left. \left. + |f'(b)|^q] \int_0^1 \left(\frac{t}{2}a + \frac{2-t}{2}b \right) dt \right)^{\frac{1}{q}} \right] \end{aligned}$$

$$= \frac{b-a}{4} [|f'(a)|^q + |f'(b)|^q]^{\frac{1}{q}} \left\{ \left(\frac{(1-\lambda)^{p+1} + \lambda^{p+1}}{p+1} \right)^{\frac{1}{p}} I_1^{\frac{1}{q}} + \left(\frac{\mu^{p+1} + (1-\mu)^{p+1}}{p+1} \right)^{\frac{1}{p}} I_2^{\frac{1}{q}} \right\} \quad (4.4)$$

We evaluate I_1 and I_2

$$I_1 = \int_0^1 \left(\frac{1+t}{2} a + \frac{1-t}{2} b \right) dt = \int_0^1 \left[\frac{1}{2} (a+b) + \frac{1}{2} (a-b)t \right] dt = \frac{3a+b}{4}$$

$$I_2 = \int_0^1 \left(\frac{t}{2} a + \frac{2-t}{2} b \right) dt = \int_0^1 \left(\frac{1}{2} (a-b) + b \right) dt = \frac{a+3b}{4}$$

We Substituting this value in 4.4

$$\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} [|f'(a)|^q + |f'(b)|^q]^{\frac{1}{q}} \left\{ \left(\frac{(1-\lambda)^{p+1} + \lambda^{p+1}}{p+1} \right)^{\frac{1}{p}} \left(\frac{3a+b}{4} \right)^{\frac{1}{q}} + \left(\frac{\mu^{p+1} + (1-\mu)^{p+1}}{p+1} \right)^{\frac{1}{p}} \left(\frac{a+3b}{4} \right)^{\frac{1}{q}} \right\} = \frac{b-a}{2} A^{\frac{1}{q}} (|f'(a)|^q + |f'(b)|^q) \left\{ \left(\frac{(1-\lambda)^{p+1} + \lambda^{p+1}}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{3a+b}{4} \right)^{\frac{1}{q}} + \left(\frac{\mu^{p+1} + (1-\mu)^{p+1}}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{a+3b}{4} \right)^{\frac{1}{q}} \right\}$$

Theorem 4.3. Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I^\circ$ with $a < b$, $q \geq 1$ and assume that $f' \in L[a, b]$. If $|f'|^q$ is a semi P-function on interval $[a, b]$, then the following inequality holds for $\lambda \geq 0, \mu \leq 1$

$$\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{24} A^{\frac{1}{q}} (|f'(a)|^q + |f'(b)|^q) \left\{ \left(\frac{1}{4} - \frac{\lambda}{2} + \frac{\lambda^2}{2} \right)^{1-\frac{1}{q}} [(4-9\lambda+12\lambda^2-2\lambda^3)a + (2-3\lambda+2\lambda^3)b]^{\frac{1}{q}} + \left(\frac{1}{4} - \frac{\mu}{2} + \frac{\mu^2}{2} \right)^{1-\frac{1}{q}} [(2-3\mu+2\mu^3)a + (4-9\mu+12\mu^2-2\mu^3)b]^{\frac{1}{q}} \right\} \quad (4.5)$$

Proof: Assume first that $q > 1$. Using the lemma 2, power mean inequality and the property of the semi P-function of $|f'|^q$ we get:

$$\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left[\left(\int_0^1 |1-\lambda-t| dt \right)^{1-\frac{1}{q}} \left(\int_0^1 |1-\lambda-t| \left| f'\left(ta + (1-t)\frac{a+b}{2} \right) \right|^q dt \right)^{\frac{1}{q}} + \left(\int_0^1 |\mu-t| dt \right)^{1-\frac{1}{q}} \left(\int_0^1 |\mu-t| \left| f'\left(t\frac{a+b}{2} + (1-t)b \right) \right|^q dt \right)^{\frac{1}{q}} \right]$$

$$\begin{aligned} &\leq \frac{b-a}{4} \left[\left(\frac{1}{2} - \lambda + \lambda^2 \right)^{1-\frac{1}{q}} \left(\int_0^1 |1-\lambda-t| \left(\frac{1+t}{2} a \right. \right. \right. \\ &\quad \left. \left. \left. + \frac{1-t}{2} b \right) [|f'(a)|^q + |f'(b)|^q] dt \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(\frac{1}{2} - \mu \right. \right. \\ &\quad \left. \left. + \mu^2 \right)^{1-\frac{1}{q}} \left(\int_0^1 |\mu-t| \left(\frac{t}{2} a \right. \right. \right. \\ &\quad \left. \left. \left. + \frac{2-t}{2} b \right) [|f'(a)|^q \right. \right. \\ &\quad \left. \left. \left. + |f'(b)|^q] dt \right)^{\frac{1}{q}} \right] \\ &\leq \frac{b-a}{4} [|f'(a)|^q \\ &\quad + |f'(b)|^q]^{\frac{1}{q}} \left[\left(\frac{1}{2} - \lambda \right. \right. \\ &\quad \left. \left. + \lambda^2 \right)^{1-\frac{1}{q}} \left(\int_0^1 |1-\lambda-t| \left(\frac{1+t}{2} a \right. \right. \right. \\ &\quad \left. \left. \left. + \frac{1-t}{2} b \right) dt \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(\frac{1}{2} - \mu \right. \right. \\ &\quad \left. \left. + \mu^2 \right)^{1-\frac{1}{q}} \left(\int_0^1 |\mu-t| \left(\frac{t}{2} a \right. \right. \right. \\ &\quad \left. \left. \left. + \frac{2-t}{2} b \right) dt \right)^{\frac{1}{q}} \right] \quad (4.6) \end{aligned}$$

We evaluate this integrals

$$\begin{aligned} &\int_0^1 |1-\lambda-t| \left(\frac{1+t}{2} a + \frac{1-t}{2} b \right) dt \\ &= \int_0^1 |1-\lambda-t| \left(\frac{1}{2} (a+b) \right. \\ &\quad \left. + \frac{1}{2} (a-b) \right) dt \\ &= \frac{1}{12} (4-9\lambda+12\lambda^2-2\lambda^3) a \\ &\quad + \frac{1}{12} (2-3\lambda+2\lambda^3) b \\ &= \frac{1}{12} [(4-9\lambda+12\lambda^2-2\lambda^3) a + (2-3\lambda+2\lambda^3) b] \end{aligned}$$

$$\begin{aligned} &\int_0^1 |\mu-t| \left(\frac{t}{2} a + \frac{2-t}{2} b \right) dt \\ &= \int_0^1 |\mu-t| \left(\frac{1}{2} (a-b)t + b \right) dt \\ &= \frac{1}{12} [(2-3\mu+2\mu^3) a \\ &\quad + (4-9\mu+12\mu^2-2\mu^3) b] \end{aligned}$$

We substitit this value in (4.6) we get

$$\begin{aligned} &= \frac{b-a}{4} [|f'(a)|^q \\ &\quad + |f'(b)|^q]^{\frac{1}{q}} \frac{1}{12} \left\{ \left(\frac{1}{2} - \lambda \right. \right. \\ &\quad \left. \left. + \lambda^2 \right)^{1-\frac{1}{q}} [(4-9\lambda+12\lambda^2-2\lambda^3) a \right. \right. \\ &\quad \left. \left. + (2-3\lambda+2\lambda^3) b]^{\frac{1}{q}} \right. \right. \\ &\quad \left. \left. + \left(\frac{1}{2} - \mu + \mu^2 \right)^{1-\frac{1}{q}} [(2-3\mu+2\mu^3) a \right. \right. \\ &\quad \left. \left. + (4-9\mu+12\mu^2-2\mu^3) b]^{\frac{1}{q}} \right\} \\ &= \frac{b-a}{24} A^{\frac{1}{q}} (|f'(a)|^q, |f'(b)|^q) \left\{ \left(\frac{1}{4} - \frac{\lambda}{2} \right. \right. \\ &\quad \left. \left. + \frac{\lambda^2}{2} \right)^{1-\frac{1}{q}} [(4-9\lambda+12\lambda^2-2\lambda^3) a \right. \right. \\ &\quad \left. \left. + (2-3\lambda+2\lambda^3) b]^{\frac{1}{q}} \right. \right. \\ &\quad \left. \left. + \left(\frac{1}{4} - \frac{\mu}{2} + \frac{\mu^2}{2} \right)^{1-\frac{1}{q}} [(2-3\mu+2\mu^3) a \right. \right. \\ &\quad \left. \left. + (4-9\mu+12\mu^2-2\mu^3) b]^{\frac{1}{q}} \right\} \end{aligned}$$

Theorem4.4. Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I^\circ$ with $a < b$, $q \geq 1$ and assume that $f' \in L[a, b]$. If $|f'|^q$ is a semi P-function on interval $[a, b]$, then the following inequality holds for $\lambda \geq 0, \mu \leq 1$

$$\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} A^{\frac{1}{q}} [|f'(a)|^q, |f'(b)|^q] \left[\frac{1}{[(q+1)(q+2)]^{\frac{1}{q}}} \{ [(q+3-\lambda)(1-\lambda)^{q+1} + (2q+4-\lambda)\lambda^{q+1}]a + [(q+1+\lambda)(1-\lambda)^{q+1} + \lambda^{q+2}]b \}^{\frac{1}{q}} + [((q+1+\mu)(1-\mu)^{q+1} + \mu^{q+2})a + [(q+3-\mu)(1-\mu)^{q+1} + (2q+4-\mu)\mu^{q+1}]b] \}^{\frac{1}{q}} \right] \quad (4.7)$$

Proof: for $q > 1$, by the property of semi P-function of $|f'|^q$ on $[a, b]$, lemma 1.1 and Holder's integral inequality, it follows that

$$\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left[\int_0^1 |1-\lambda - t| \left| f'\left(ta + (1-t)\frac{a+b}{2}\right) \right| dt + \int_0^1 |\mu - t| \left| f'\left(t\frac{a+b}{2} + (1-t)b\right) \right| dt \right] \leq \left[\left(\int_0^1 dt \right)^{1-\frac{1}{q}} \left(\int_0^1 |1-\lambda - t|^q \left(\frac{1+t}{2}a + \frac{1-t}{2}b \right) [|f'(a)|^q + |f'(b)|^q] dt \right)^{\frac{1}{q}} + \left(\int_0^1 dt \right)^{1-\frac{1}{q}} \left(\int_0^1 |\mu - t|^q \left(\frac{t}{2}a + \frac{2-t}{2}b \right) [|f'(a)|^q + |f'(b)|^q] dt \right)^{\frac{1}{q}} \right]$$

$$= \frac{b-a}{4} [|f'(a)|^q + |f'(b)|^q]^{\frac{1}{q}} \left(\int_0^1 |1-\lambda - t|^q \left(\frac{1+t}{2}a + \frac{1-t}{2}b \right) dt + \int_0^1 |\mu - t|^q \left(\frac{t}{2}a + \frac{2-t}{2}b \right) dt \right)^{\frac{1}{q}} \quad (4.8)$$

By lemma 2.2, we have

$$\begin{aligned} & \int_0^1 |1-\lambda - t|^q \left(\frac{1+t}{2}a + \frac{1-t}{2}b \right) dt \\ &= \int_0^1 |1-\lambda - t|^q \left(\frac{1}{2}(a+b) + \frac{1}{2}(a-b) \right) dt \\ &= \frac{1}{2}(a+b) \int_0^1 |1-\lambda - t|^q dt + \frac{1}{2}(a-b) \int_0^1 t |1-\lambda - t|^q dt \\ &= \frac{1}{2}(a+b) \frac{\lambda^{q+1} + (1-\lambda)^{q+1}}{(q+1)} + \frac{1}{2}(a-b) \frac{(1-\lambda)^{q+2} + (q+2-\lambda)\lambda^{q+1}}{(q+1)(q+2)} \\ &= \frac{(a+b)}{2(q+1)} [\lambda^{q+1} + (1-\lambda)^{q+1}] + \frac{(a-b)}{2(q+1)(q+2)} [(1-\lambda)^{q+2} + (q+2-\lambda)\lambda^{q+1}] \\ &= \frac{1}{2(q+1)(q+2)} \{ [(q+3-\lambda)(1-\lambda)^{q+1} + (2q+4-\lambda)\lambda^{q+1}]a + [(q+1+\lambda)(1-\lambda)^{q+1} + \lambda^{q+2}]b \} \\ & \quad \int_0^1 |\mu - t|^q \left(\frac{t}{2}a + \frac{2-t}{2}b \right) dt \\ &= \frac{1}{2(q+1)(q+2)} \{ [(q+1+\mu)(1-\mu)^{q+1} + \mu^{q+2}]a + [(q+3-\mu)(1-\mu)^{q+1} + (2q+4-\mu)\mu^{q+1}]b \} \end{aligned}$$

Substitute this in (4.8)

$$\begin{aligned} & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) \right. \\ & \quad \left. - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4} [|f'(a)|^q \\ & \quad + |f'(b)|^q]^{\frac{1}{q}} \left[\frac{1}{2(q+1)(q+2)} \right]^{\frac{1}{q}} \left\{ [(q+3-\lambda)(1-\lambda)^{q+1} \right. \\ & \quad + (2q+4-\lambda)\lambda^{q+1}]a \\ & \quad + [(q+1+\lambda)(1-\lambda)^{q+1} + \lambda^{q+2}]b^{\frac{1}{q}} \\ & \quad + [(q+1+\mu)(1-\mu)^{q+1} + \mu^{q+2}]a \\ & \quad + [(q+3-\mu)(1-\mu)^{q+1} \\ & \quad + (2q+4-\mu)\mu^{q+1}]b^{\frac{1}{q}} \Big\} \\ & = \frac{b-a}{4} A^{\frac{1}{q}} [|f'(a)|^q, |f'(b)|^q] \left[\frac{1}{(q+1)(q+2)} \right]^{\frac{1}{q}} \left\{ [(q+3-\lambda)(1-\lambda)^{q+1} \right. \\ & \quad + (2q+4-\lambda)\lambda^{q+1}]a \\ & \quad + [(q+1+\lambda)(1-\lambda)^{q+1} + \lambda^{q+2}]b^{\frac{1}{q}} \\ & \quad + [(q+1+\mu)(1-\mu)^{q+1} + \mu^{q+2}]a \\ & \quad + [(q+3-\mu)(1-\mu)^{q+1} \\ & \quad + (2q+4-\mu)\mu^{q+1}]b^{\frac{1}{q}} \Big\} \end{aligned}$$

Corollary 4.1. Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I^\circ$ with $a < b$, $0 \leq \lambda \leq 1$, and $f' \in L[a, b]$. If $|f'|^q$ is a semi P-function for $q \geq 1$ on $[a, b]$, then

$$\begin{aligned} & \left| \lambda \frac{f(a) + f(b)}{2} + (1-\lambda) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4} A^{\frac{1}{q}} [|f'(a)|^q, |f'(b)|^q] \left[\frac{1}{(q+1)(q+2)} \right]^{\frac{1}{q}} \left\{ [(q+3-\lambda)(1-\lambda)^{q+1} \right. \\ & \quad + (2q+4-\lambda)\lambda^{q+1}]a \\ & \quad + [(q+1+\lambda)(1-\lambda)^{q+1} + \lambda^{q+2}]b^{\frac{1}{q}} \\ & \quad + [(q+1+\lambda)(1-\lambda)^{q+1} + \lambda^{q+2}]a \\ & \quad + [(q+3-\lambda)(1-\lambda)^{q+1} \\ & \quad + (2q+4-\lambda)\lambda^{q+1}]b^{\frac{1}{q}} \Big\} \quad (4.9) \end{aligned}$$

Corollary 4.2. Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I^\circ$ with $a < b$, and $f' \in L[a, b]$. If $|f'|^q$ is a semi P-function for $q \geq 1$ on $[a, b]$, then

$$\begin{aligned} & \left| \frac{1}{2} \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{8} A^{\frac{1}{q}} [|f'(a)|^q, |f'(b)|^q] \left[\frac{1}{2(q+1)(q+2)} \right]^{\frac{1}{q}} \left\{ [(3q+6)a + (q+2)b]^{\frac{1}{q}} \right. \\ & \quad \left. + [(q+2)a + (3q+6)b]^{\frac{1}{q}} \right\} \quad (4.10) \end{aligned}$$

Corollary 4.3. Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on I° , $a, b \in I^\circ$ with $a < b$, and $f' \in L[a, b]$. If $|f'|^q$ is a semi P-function for $q \geq 1$ on $[a, b]$ and

$$\frac{f(a) + f(b)}{2} = f\left(\frac{a+b}{2}\right)$$

Then

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & = \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{16} A^{\frac{1}{q}} [|f'(a)|^q, |f'(b)|^q] \left[\frac{1}{2(q+1)(q+2)} \right]^{\frac{1}{q}} \left\{ [(2q+5)a + (2q+3)b]^{\frac{1}{q}} \right. \\ & \quad + [a + (4q+7)b]^{\frac{1}{q}} \\ & \quad + [(4q+7)a + b]^{\frac{1}{q}} \\ & \quad \left. + [(2q+3)a + (2q+5)b]^{\frac{1}{q}} \right\} \quad (4.11) \end{aligned}$$

Theorem 4.5. Let $a, b > 0$, $q \geq 1$, either $s > 1$ and $(s-1)q \geq 1$ or $s < 0$, then

$$\begin{aligned} & \left| \frac{A(a^s, b^s) + A^s(a, b)}{2} - L^s(a, b) \right| \\ & \leq \frac{b-a}{8} |s| \left[\frac{a^{(s-1)q} + b^{(s-1)q}}{4(q+1)(q+2)} \right]^{\frac{1}{q}} \left\{ [(3q+6)a + (q+2)b]^{\frac{1}{q}} \right. \\ & \quad \left. + [(q+2)a + (3q+6)b]^{\frac{1}{q}} \right\} \quad (4.12) \end{aligned}$$

Proof: we use the corollary 4.2 and we get

$$\begin{aligned} & \left| \frac{1}{2} \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{8} A^{\frac{1}{q}} [|f'(a)|^q, |f'(b)|^q] \left[\frac{1}{2(q+1)(q+2)} \right]^{\frac{1}{q}} \left\{ [(3q+6)a + (q+2)b]^{\frac{1}{q}} \right. \\ & \quad \left. + [(q+2)a + (3q+6)b]^{\frac{1}{q}} \right\} \end{aligned}$$

By taking $f(x) = x^s$ in corollary 4.2 we get

$$\begin{aligned} f(a) = a^s, f(b) = b^s & \Rightarrow \frac{f(a) + f(b)}{2} = \frac{a^s + b^s}{2} \\ & = A(a^s, b^s) \\ f\left(\frac{a+b}{2}\right) & = \left(\frac{a+b}{2}\right)^s = A^s(a, b) \end{aligned}$$

$$\begin{aligned}\frac{1}{b-a} \int_a^b f(x) dx &= \frac{1}{b-a} \int_a^b x^s dx = \frac{1}{b-a} \left[\frac{x^{s+1}}{s+1} \right]_a^b \\ &= \frac{1}{b-a} \left[\frac{b^{s+1}}{s+1} - \frac{a^{s+1}}{s+1} \right] \\ &= \frac{1}{b-a} \left[\frac{b^{s+1} - a^{s+1}}{s+1} \right] \\ &= \frac{b^{s+1} - a^{s+1}}{(b-a)(s+1)} = L^s(a, b)\end{aligned}$$

The left hand inequality has the form of below

$$\begin{aligned}\left| \frac{1}{2} \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ = \left| \frac{1}{2} [A(a^s, b^s) + A^s(a, b)] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ = \left| \frac{A(a^s, b^s) + A^s(a, b)}{2} - L^s(a, b) \right|\end{aligned}$$

The left hand inequality has the form of below

$$\begin{aligned}f(x) = x^s &\Rightarrow f'(x) = sx^{s-1} \Rightarrow f'(a) = sa^{s-1} \\ f(x) = x^s &\Rightarrow f'(x) = sx^{s-1} \Rightarrow f'(b) = sb^{s-1} \\ A^{\frac{1}{q}}[|f'(a)|^q, |f'(b)|^q] &= \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \\ &= \left[\frac{|s|^q a^{(s-1)q} + |s|^q b^{(s-1)q}}{2} \right]^{\frac{1}{q}} \\ &= |s| \left[\frac{a^{(s-1)q} + b^{(s-1)q}}{2} \right]^{\frac{1}{q}}\end{aligned}$$

We substitute this value in the right inequality

$$\begin{aligned}\frac{b-a}{8} A^{\frac{1}{q}}[|f'(a)|^q, |f'(b)|^q] \left[\frac{1}{2(q+1)(q+2)} \right]^{\frac{1}{q}} \left\{ [(3q+6)a + (q+2)b]^{\frac{1}{q}} \right. \\ \left. + [(q+2)a + (3q+6)b]^{\frac{1}{q}} \right\}\end{aligned}$$

In conclusion we get

$$\begin{aligned}\frac{b-a}{8} |s| \left[\frac{a^{(s-1)q} + b^{(s-1)q}}{4(q+1)(q+2)} \right]^{\frac{1}{q}} \left\{ [(3q+6)a + (q+2)b]^{\frac{1}{q}} \right. \\ \left. + [(q+2)a + (3q+6)b]^{\frac{1}{q}} \right\}\end{aligned}$$

Thus

$$\begin{aligned}\left| \frac{A(a^s, b^s) + A^s(a, b)}{2} - L^s(a, b) \right| \\ \leq \frac{b-a}{8} |s| \left[\frac{a^{(s-1)q} + b^{(s-1)q}}{4(q+1)(q+2)} \right]^{\frac{1}{q}} \left\{ [(3q+6)a + (q+2)b]^{\frac{1}{q}} \right. \\ \left. + [(q+2)a + (3q+6)b]^{\frac{1}{q}} \right\}\end{aligned}$$

V. CONCLUSION

In this paper, several extended Hermite–Hadamard type inequalities for semi P-functions have been investigated. By using the properties of semi P-functions, extended Hermite–Hadamard inequalities were established to obtain integral bounds under weaker convexity conditions. Furthermore, Hölder's inequality and the power mean inequality were employed to derive additional estimates and refinements of the obtained results.

The results show that the use of Hölder's inequality and the power mean inequality provides different forms of upper and lower bounds for the corresponding integral expressions. These inequalities extend the applicability of Hermite–Hadamard type results to the class of semi-(P)-functions and provide several estimates that can be useful in the study of integral inequalities.

Moreover, some applications of the obtained inequalities to classical means were presented. These applications demonstrate how the derived results can be used to establish bounds and relationships involving different means. Thus, the obtained inequalities provide a useful framework for connecting generalized convexity concepts with integral inequalities and mean inequalities. The results obtained in this study may also provide a basis for further investigations involving other generalized convexity classes, different integral operators, and alternative inequalities such as Jensen-type, Simpson-type, and Ostrowski-type inequalities.

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