

Application of the Natural Transform Decomposition Method to Fractional-Order KdV Equations

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ABSTRACT

In this paper, the analytical solutions of fractional-order Korteweg–de Vries (KdV) equations are investigated by applying the Natural Transform Decomposition Method (NTDM). The proposed method combines the Natural Transform with the Adomian decomposition technique to obtain approximate series solutions of linear and nonlinear dispersive fractional partial differential equations. Several illustrative examples of fractional KdV equations are considered and solved systematically using NTDM. The Caputo fractional derivative is employed to describe the fractional-order behavior of the models. The obtained solutions are expressed in rapidly convergent series forms and compared with the corresponding exact solutions in the integer-order case. The results demonstrate that the proposed method is accurate, efficient, and computationally simple for solving fractional KdV equations without requiring discretization, perturbation, or linearization. Furthermore, it is observed that the fractional-order solutions converge to the classical exact solutions as the fractional parameter approaches unity. The study confirms that NTDM is a reliable analytical technique for investigating nonlinear fractional dispersive equations arising in mathematical physics and engineering sciences.

Keywords- Natural transform, Dispersive equations, Fractional KdV equations, Adomian Decomposition.

I. INTRODUCTION

Fractional calculus, which generalizes the concepts of differentiation and integration to non-integer orders, has emerged as a powerful mathematical tool for modeling complex phenomena in science and engineering. Unlike classical integer-order differential equations, fractional differential equations incorporate memory and hereditary properties, enabling a more accurate description of many physical, biological, and engineering systems. During the last few decades, significant attention has been devoted to the theoretical development and practical applications of fractional calculus. Fundamental contributions to this field can be

found in the works of Podlubny [1], Kilbas et al. [2], Oldham and Spanier [3], and Mainardi [4]. The introduction of the Caputo fractional derivative [5] further enhanced the applicability of fractional models by allowing the use of physically meaningful initial conditions.

Many nonlinear phenomena occurring in fluid dynamics, plasma physics, optical fibers, and shallow water waves are described by nonlinear partial differential equations. Among these equations, the Korteweg–de Vries (KdV) equation occupies a central position due to its ability to model the propagation of weakly nonlinear and weakly dispersive waves. The classical KdV equation has been extensively studied and possesses rich

mathematical structures, including soliton solutions and conservation laws. To capture memory effects and anomalous transport processes that arise in real-world applications, researchers have extended the classical KdV equation to its fractional-order counterpart. Fractional KdV equations provide more realistic mathematical models for various physical systems where nonlocal interactions and history-dependent processes play significant roles.

Obtaining exact or approximate analytical solutions of nonlinear fractional differential equations remains a challenging task because of their inherent nonlinearity and the presence of fractional derivatives. Consequently, numerous analytical and semi-analytical methods have been developed to investigate such equations. Among the most widely used techniques are the Variational Iteration Method (VIM) introduced by He [6], the Homotopy Perturbation Method (HPM) proposed by Momani and Odibat [7], the Adomian Decomposition Method (ADM) developed by Adomian [8], and the Homotopy Analysis Method (HAM) introduced by Liao [10]. These methods have demonstrated considerable success in solving a variety of linear and nonlinear fractional differential equations. Furthermore, Jafari and Daftardar-Gejji [9] employed decomposition techniques for systems of nonlinear fractional differential equations, illustrating the effectiveness of such approaches in handling nonlinear terms.

Integral transform methods have also become valuable tools for solving differential equations. In particular, the Natural Transform, introduced and developed by Belgacem and co-authors [11,13], combines useful properties of both the Laplace and Sumudu transforms and provides an efficient framework for analyzing linear and nonlinear problems. The transform simplifies the treatment of differential equations and often yields rapidly convergent series solutions. Motivated by these advantages, researchers have combined the Natural Transform with decomposition techniques to construct the Natural Transform Decomposition Method (NTDM), which effectively addresses nonlinear fractional differential equations without requiring linearization, discretization, or restrictive assumptions.

Several studies have demonstrated the applicability of Natural Transform-based methods. Khalouta and Kadem [12] developed the Fractional Natural Decomposition Method for a class of nonlinear time-fractional wave-like equations with variable coefficients and obtained accurate analytical approximations. Rawashdeh and Al-Jammal [14] employed the Fractional Natural Decomposition Method to solve systems of nonlinear fractional ordinary differential equations, showing that the method produces reliable and rapidly convergent solutions. Related transform-decomposition approaches have also been investigated for nonlinear integro-fractional differential equations and other classes of fractional models [15]. Moreover, convergence analyses of iterative and semi-

analytical methods have been studied by several authors, including Odibat [16] and Zhang et al. [17], providing theoretical support for the effectiveness of such techniques.

Motivated by the growing importance of fractional nonlinear wave equations and the effectiveness of transform-based decomposition methods, the present work applies the Natural Transform Decomposition Method to obtain analytical solutions of fractional-order KdV equations. The proposed approach combines the operational advantages of the Natural Transform with the recursive structure of the decomposition method, leading to solution series that are straightforward to construct and computationally efficient. The method avoids perturbation parameters, linearization procedures, and complicated numerical discretizations while maintaining high accuracy.

The main objective of this paper is to demonstrate the applicability and efficiency of NTDM in solving fractional-order KdV equations. Several illustrative examples are considered, and the resulting analytical solutions are obtained in series form. The obtained results confirm that NTDM provides accurate approximations and converges rapidly to the exact solutions whenever they are available. Therefore, the method represents a reliable and effective analytical technique for studying nonlinear fractional evolution equations arising in mathematical physics and engineering applications.

II. PRELIMINARIES AND BASIC CONCEPTS

In this section, we define a few terms and provide some fundamental details about the Natural Transform Decomposition Method.

Definition 2.1. A real function $g(x)$, $x > 0$ is said to be in space C_μ ($\mu \in \mathbb{R}$) if there exists a real number $p > \mu$, such that $g(x) = x^p g_1(x)$, where $g_1(x) \in C(0, \infty)$, and it is said to be in space C_μ^n if $f^n \in C_\mu^n$, $n \in \mathbb{N}$ [17].

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in C_\mu$, $\mu \geq -1$, is defined as [17]:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} dt, \alpha > 0, t > 0,$$

$$J^0 f(x) = f(x).$$

For $f(x) \in C_\mu$, where $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma \geq -1$,

some properties of the operator J^α are given as follows:

$$(i) J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x), \quad (ii) J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x),$$

$$(iii) J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}.$$

Definition 2.3. The fractional derivative of $f(x)$ in the Caputo sense is defined as [17]:

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^n(t) dt,$$

For $-1 < \alpha \leq m$, $m \in \mathbb{N}$, $t > 0$, $f \in C_{-1}^m$.

Lemma 2.1. If $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$, $f \in C_\mu$, $\mu \geq -1$, then the following two properties hold [17]:

(i) $D^\alpha [J^\alpha f(x)] = f(x)$;

(ii) $J^\alpha [D^\alpha f(x)] = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{x^k}{k!}$

Definition 2.4. The Natural transform of a function $f(t) \in \mathbb{R}^2$ is given by the following integral equation [13].

$$\mathbb{N}^+[f(t)] = \int_0^\infty e^{-st} f(ut) dt, \operatorname{Re}(s) > 0, u \in (-\tau_1, \tau_2)$$

The above mentioned equation is strictly converges to the Laplace transform $F(s)$ when $u \equiv 1$. Some properties of Natural transform which are needed here are as follows:

2.5.1. If a & b are any constants and $f(t)$ and $g(t)$ are functions, then

$$\mathbb{N}^+[af(t) + bg(t)] = a\mathbb{N}^+[f(t)] + b\mathbb{N}^+[g(t)]$$

2.5.2. If $\xi^n(t)$ is the n -th derivative of function $\xi(t)$ then the natural transform of n -th derivative is given by:

$$\begin{aligned} \mathbb{N}^+[\xi^n(t)] &= \mathbb{Q}_n(s, u) \\ &= \frac{s^n}{u^n} \mathbb{Q}(s, u) \\ &\quad - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{u^{n-k}} g^n(0), n \\ &\geq 1 \end{aligned}$$

Definition 2.5. The inverse Natural transform of a function is defined by [13]:

$$\mathbb{N}^-[R(s, u)] = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{st}{u}} R(s, u) ds$$

Definition 2.6. The Natural transform of Mittag-Leffler function $E_{\alpha, \beta}$ is defined as follows [18]:

$$\mathbb{N}^+[f(x, t)] = \int_0^\infty e^{-st} f(x, ut) dt = \sum_{k=0}^\infty \frac{u^{k+1} \Gamma(k + \beta)}{s^{k+1} \Gamma(\alpha k + \beta)}$$

Definition 2.7. A two parameter function of Mittag-Leffler type is defined by series Expansion [18].

$$E_{\alpha, \beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \alpha, \beta > 0$$

III. NATURAL TRANSFORM DECOMPOSITION METHOD

In this section NTDM is implemented to solve fractional dispersive PDE of order three.

$$D_t^\alpha \psi(x, t) + \varsigma \frac{\partial^3 \psi(x, t)}{\partial x^3} = \xi(x, t), \quad \varsigma, t \geq 0, \quad m - 1 < \alpha < m, \quad (1)$$

$\xi(x, t)$ Is the source function.

Subject to initial condition.

$$\psi(x, 0) = p(x) \quad (2)$$

By implementing the Natural transform, Equation (1) can be written as:

$$\mathbb{N}^+[D_t^\alpha \psi(x, t)] + \mathbb{N}^+\left[\varsigma \frac{\partial^3 \psi(x, t)}{\partial x^3}\right] = \mathbb{N}^+[\xi(x, t)], \quad (3)$$

$$\begin{aligned} \frac{s^\alpha}{u^\alpha} \mathbb{N}^+[\psi(x, t)] - \frac{s^{\alpha-1}}{u^\alpha} \psi(x, 0) \\ = \mathbb{N}^+[\xi(x, t)] \\ - \mathbb{N}^+\left[\varsigma \frac{\partial^3 \psi(x, t)}{\partial x^3}\right], \quad (4) \end{aligned}$$

$$\begin{aligned} \mathbb{N}^+[\psi(x, t)] = \frac{p(x)}{s} + \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[\xi(x, t)] - \\ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+\left[\varsigma \frac{\partial^3 \psi(x, t)}{\partial x^3}\right]. \quad (5) \end{aligned}$$

After applying the inverse of Natural transform to both side of equation (5), the result can be written as:

$$\begin{aligned} \psi(x, t) = \mathbb{N}^- \left[\frac{p(x)}{s} \right] + \mathbb{N}^- \left[\frac{u^\alpha}{s^\alpha} \mathbb{N}^+[\xi(x, t)] \right] \\ - \mathbb{N}^- \left[\frac{u^\alpha}{s^\alpha} \mathbb{N}^+ \left[\varsigma \frac{\partial^3 \psi(x, t)}{\partial x^3} \right] \right]. \quad (6) \end{aligned}$$

The NTDM solution $\psi(x, t)$ is represented by the following infinite series.

$$\psi(x, t) = \sum_{l=0}^\infty \psi_l(x, t). \quad (7)$$

And the nonlinear term (if any) in problem are defined by infinite series of Adomian polynomials.

$$\begin{aligned} N\psi(x, t) \\ = \sum_{n=0}^\infty A_n, \quad (8) \end{aligned}$$

$$\begin{aligned} A_n = \frac{1}{n!} \left[\frac{\partial^n}{\partial \xi^n} \left[F \sum_{n=0}^\infty \omega_n \xi^n \right] \right]_{\xi=0} n \\ = 0, 1, 2, \dots, \quad (9) \end{aligned}$$

After computing every term of equation (7), we arrive at

$$\begin{aligned} \psi(x, t) = \psi_0(x, t) + \psi_1(x, t) + \psi_2(x, t) + \dots \\ = \sum_{l=0}^\infty \psi_l(x, t) \quad (10) \end{aligned}$$

The NTDM procedure described above is formulated for the one-dimensional case. Its extension to higher-dimensional problems follows the same methodology. Interested readers are referred to [18] for a detailed discussion.

IV. ILLUSTRATIVE EXAMPLES

In this section, for further clarification, we consider several numerical examples.

Example 1: Consider the following fractional dispersive KdV equation [18].

$$\begin{aligned} D_t^\alpha \psi(x, t) + 2\psi_x + \psi_{xxx} = 0, \\ t > 0, 0 < \alpha \\ \leq 1, \quad (11) \end{aligned}$$

With the initial condition

$$\begin{aligned} \psi(x, 0) \\ = \sin x. \quad (12) \end{aligned}$$

Solution:

Apply the Natural Transform to both sides of the equation (11).

$$\mathbb{N}^+[D_t^\alpha \psi(x, t)] + \mathbb{N}^+[2\psi_x] + \mathbb{N}^+[\psi_{xxx}] = 0,$$

$$\frac{s^\alpha}{u^\alpha} \mathbb{N}^+[\psi(x, t)] - \frac{s^{\alpha-1}}{u^\alpha} \psi(x, 0) = \mathbb{N}^+[-2\psi_x - \psi_{xxx}],$$

$$\mathbb{N}^+[\psi(x, t)] = \frac{\sin x}{s} + \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[-2\psi_x - \psi_{xxx}]. \quad (13)$$

Apply the inverse Natural Transform to the equation (13).

$$\psi(x, t) = \mathbb{N}^- \left[\frac{\sin x}{s} \right] + \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[-2\psi_x - \psi_{xxx}] \right\},$$

$$\psi_0(x, t) = \mathbb{N}^- \left[\frac{\sin x}{s} \right] = \sin x \mathbb{N}^- \left[\frac{1}{s} \right] = \sin x. \quad (14)$$

$$\psi_{l+1}(x, t) = \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[-2\psi_{lx} - \psi_{lxxx}] \right\}. \quad (15)$$

For $l = 0, 1, 2, \dots$

$$\psi_1(x, t) = \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[-2\psi_{0x} - \psi_{0xxx}] \right\}.$$

Now, for $l = 0$ we compute each term:

$$\psi_{0x} = \frac{\partial}{\partial x} (\psi_0(x, t)) = \frac{\partial}{\partial x} (\sin x) = \cos x,$$

$$\psi_{0xxx} = \frac{\partial^3}{\partial x^3} \psi_0(x, t) = -\cos x,$$

$$\psi_1(x, t) = \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[-2\cos x + \cos x] \right\} = \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[-\cos x] \right\} = -\cos x \mathbb{N}^- \left\{ \frac{u^\alpha}{s^{\alpha+1}} \right\},$$

$$\psi_1(x, t) = -\cos x \frac{t^\alpha}{\Gamma(\alpha + 1)}, \quad (16)$$

$l = 1$

$$\psi_2(x, t) = \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[-2\psi_{1x} - \psi_{1xxx}] \right\}. \quad (17)$$

Now, we compute each term:

$$\psi_{1x} = \frac{\partial}{\partial x} (\psi_1(x, t)) = \sin x \frac{t^\alpha}{\Gamma(\alpha + 1)},$$

$$\psi_{1xxx} = \frac{\partial^3}{\partial x^3} \psi_1(x, t) = -\sin x \frac{t^\alpha}{\Gamma(\alpha + 1)},$$

$$\psi_2(x, t) = -\sin x \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+ \left[\frac{t^\alpha}{\Gamma(\alpha + 1)} \right] \right\},$$

$$\psi_2(x, t) = -\sin x \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}, \quad (18)$$

The NTDM solution for example 1 is

$$\psi(x, t) = \psi_0(x, t) + \psi_1(x, t) + \psi_2(x, t) + \dots$$

$$= \sum_{i=0}^n \psi_i(x, t), \quad (19)$$

$$\psi(x, t) = \sin x - \cos x \frac{t^\alpha}{\Gamma(\alpha + 1)} - \sin x \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \dots \quad (20)$$

For $\alpha = 1$, the exact form of $\psi(x, t)$ is

$$\psi(x, t) = \sin x - \frac{t}{1!} \cos x - \frac{t^2}{2!} \sin x + \frac{t^3}{3!} \cos x - \dots$$

$$= \sin(x - t), \quad (21)$$

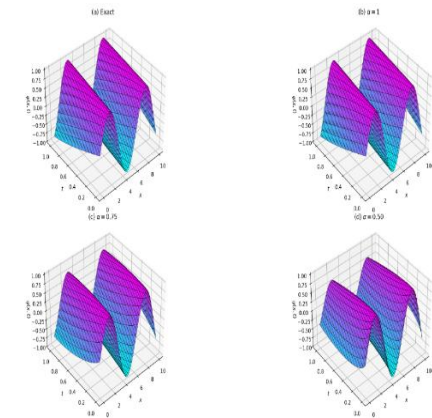


Figure 1: The (a) Exact and NTDM solutions of $\psi(x, t)$ of example 1 at (b) $\alpha = 1$, (c) $\alpha = 0.75$, and (d) $\alpha = 0.50$.

Table 1. Absolute error for Example 1 at $t = 0.2$.

	Exact-NTDM	Exact-NTDM	Exact-NTDM
x	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.8$
0.1	0.2931	0.1268	0.0335
0.2	0.2902	0.1252	0.0329
0.3	0.2845	0.1223	0.0319
0.4	0.2759	0.1182	0.0304
0.5	0.2646	0.1129	0.0285
0.6	0.2506	0.1065	0.0263
0.7	0.2341	0.0990	0.0238
0.8	0.2153	0.0905	0.0209
0.9	0.1943	0.0811	0.0178
1.0	0.1714	0.0709	0.0145

Example 2: Consider the following fractional dispersive KdV in equation [18].

$$D_t^\alpha \psi(x, y, t) + \psi_{xxx} + \psi_{yyy} = 0, \quad t > 0, \quad 0 < \alpha \leq 1, \quad (22)$$

With the initial condition

$$\psi(x, y, 0) = \cos(x + y). \quad (23)$$

Solution:

Applying the Natural Transform to both sides of equation (22) yields

$$\mathbb{N}^+[D_t^\alpha \psi(x, y, t)] + \mathbb{N}^+[\psi_{xxx}] + \mathbb{N}^+[\psi_{yyy}] = 0,$$

$$\frac{s^\alpha}{u^\alpha} \mathbb{N}^+[\psi(x, y, t)] - \frac{s^{\alpha-1}}{u^\alpha} \psi(x, y, 0) = \mathbb{N}^+[-\psi_{xxx} - \psi_{yyy}],$$

$$\frac{s^\alpha}{u^\alpha} \mathbb{N}^+[\psi(x, y, t)] = \frac{s^{\alpha-1}}{u^\alpha} \cos(x + y) + \mathbb{N}^+[-\psi_{xxx} - \psi_{yyy}] \quad (24)$$

$$\mathbb{N}^+[\psi(x, y, t)] = \frac{\cos(x + y)}{s} + \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[-\psi_{xxx} - \psi_{yyy}], \quad (25)$$

Apply the inverse of natural transform to the equation (25).

$$\psi(x, y, t) = \mathbb{N}^- \left[\frac{\cos(x+y)}{s} \right] + \mathbb{N}^- \left[\frac{u^\alpha}{s^\alpha} \mathbb{N}^+ [-\psi_{xxx} - \psi_{yyy}] \right],$$

$$\psi_0(x, y, t) = \mathbb{N}^- \left[\frac{\cos(x+y)}{s} \right] = \cos(x+y),$$

$$\psi_{l+1}(x, y, t) = \mathbb{N}^- \left[\frac{u^\alpha}{s^\alpha} \mathbb{N}^+ [-\psi_{lxxx} - \psi_{lyyy}] \right]. \quad (26)$$

For $l = 0, 1, 2, \dots$

$$\psi_1(x, y, t) = \mathbb{N}^- \left[\frac{u^\alpha}{s^\alpha} \mathbb{N}^+ [-\psi_{0xxx} - \psi_{0yyy}] \right]. \quad (27)$$

Now, for $l = 0$ we compute each term:

$$\psi_{0xxx} = \frac{\partial^3}{\partial x^3} (\psi_0(x, y, t)) = \sin(x+y),$$

$$\psi_{0yyy} = \frac{\partial^3}{\partial y^3} (\psi_0(x, y, t)) = \sin(x+y),$$

$$\psi_1(x, y, t) = \mathbb{N}^- \left[\frac{u^\alpha}{s^\alpha} \mathbb{N}^+ [-\sin(x+y) - \sin(x+y)] \right]$$

$$= \mathbb{N}^- \left[\frac{u^\alpha}{s^\alpha} \mathbb{N}^+ [-2\sin(x+y)] \right]$$

$$= \mathbb{N}^- \left[-2\sin(x+y) \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[1] \right]$$

$$= -2\sin(x+y) \mathbb{N}^- \left[\frac{u^\alpha}{s^{\alpha+1}} \right],$$

$$\psi_1(x, y, t) = -2\sin(x+y) \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad (28)$$

$$\psi_2(x, y, t) = \mathbb{N}^- \left[\frac{u^\alpha}{s^\alpha} \mathbb{N}^+ [-\psi_{1xxx} - \psi_{1yyy}] \right] \quad (29)$$

$l = 1$

Now we compute each term:

$$\psi_{1xxx} = \frac{\partial^3}{\partial x^3} [\psi_1(x, y, t)] = 2 \cos(x+y) \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$\psi_{1yyy} = \frac{\partial^3}{\partial y^3} [\psi_1(x, y, t)] = 2 \cos(x+y) \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$\psi_2(x, y, t) = \mathbb{N}^- \left[\frac{u^\alpha}{s^\alpha} \mathbb{N}^+ \left[-2 \cos(x+y) \frac{t^\alpha}{\Gamma(\alpha+1)} - 2 \cos(x+y) \frac{t^\alpha}{\Gamma(\alpha+1)} \right] \right]$$

$$\psi_2(x, y, t) = \mathbb{N}^- \left[\frac{u^\alpha}{s^\alpha} \mathbb{N}^+ \left[-4 \cos(x+y) \frac{t^\alpha}{\Gamma(\alpha+1)} \right] \right]$$

$$= -4 \cos(x+y) \mathbb{N}^- \left[\frac{u^{2\alpha}}{s^{2\alpha}} \right]$$

$$\psi_2(x, y, t) = -4 \cos(x+y) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \quad (30)$$

The NTDM solution for example 2 is

$$\psi(x, y, t) = \psi_0(x, y, t) + \psi_1(x, y, t) + \psi_2(x, y, t) + \dots$$

$$= \sum_{i=0}^n \psi_i(x, y, t). \quad (31)$$

$$\psi(x, y, t) = \cos(x+y) - 2\sin(x+y) \frac{t^\alpha}{\Gamma(\alpha+1)} - 4 \cos(x+y) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \dots \quad (32)$$

For $\alpha = 1$ the exact form of $\psi(x, y, t)$ is

$$\psi(x, y, t) = \cos(x+y) - 2 \sin(x+y) \frac{t}{1!} - 4 \cos(x+y) \frac{t^2}{2!} + 8 \sin(x+y) \frac{t^3}{3!} + \dots$$

$$\psi(x, y, t) = \cos(x+y + 2t). \quad (33)$$

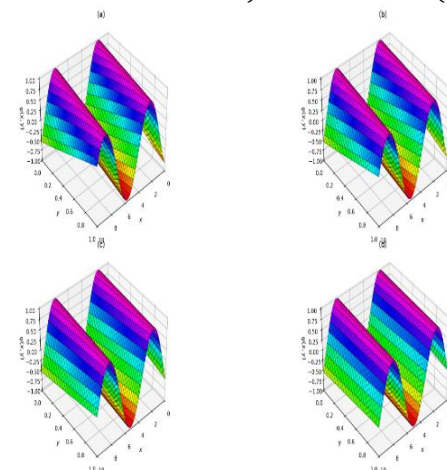


Figure 2: The (a) Exact and NTDM solutions of $\psi(x, y, t)$ of example 2 at (b) $\alpha = 1$, (c) $\alpha = 0.75$, and (d) $\alpha = 0.50$.

Table 2. Absolute error for Example 2 at $t = 0.2$ and $y = 0.5$.

	Exact-NTDM	Exact-NTDM	Exact-NTDM
x	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.8$
0.1	0.3753	0.1628	0.0448
0.2	0.3787	0.1636	0.0449
0.3	0.3784	0.1628	0.0445
0.4	0.3743	0.1604	0.0437
0.5	0.3664	0.1563	0.0424
0.6	0.3549	0.1507	0.0407
0.7	0.3398	0.1435	0.0386
0.8	0.3213	0.1349	0.0361
0.9	0.2995	0.1249	0.0332
1.0	0.2747	0.1136	0.0300

Example 3: Consider the following non-homogeneous fractional dispersive KdV in equation [18].

$$D_t^\alpha \psi(x, t) + \psi_{xxx} = -\sin \pi x \sin t - \pi^3 \cos \pi x \cos t, \quad 0 < \alpha \leq 1, \quad (34)$$

With the initial condition

$$\psi(x, 0) = \sin \pi x. \quad (35)$$

Solution:

Apply the natural transform to the both side of equation (34).

$$\begin{aligned} \mathbb{N}^+[D_t^\alpha \psi(x, t)] + \mathbb{N}^+[\psi_{xxx}] &= \mathbb{N}^+[-\sin \pi x \sin t - \pi^3 \cos \pi x \cos t], \\ \frac{s^\alpha}{u^\alpha} \mathbb{N}^+[\psi(x, t)] - \frac{s^{\alpha-1}}{u^\alpha} \psi(x, 0) &= -\mathbb{N}^+[\psi_{xxx}] \\ &+ \mathbb{N}^+[-\sin \pi x \sin t - \pi^3 \cos \pi x \cos t], \\ \frac{s^\alpha}{u^\alpha} \mathbb{N}^+[\psi(x, t)] &= \frac{s^{\alpha-1}}{u^\alpha} \sin \pi x - \mathbb{N}^+[\psi_{xxx} \\ &+ \sin \pi x \sin t + \pi^3 \cos \pi x \cos t], \\ \mathbb{N}^+[\psi(x, t)] &= \frac{\sin \pi x}{s} - \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[\sin \pi x \sin t \\ &+ \pi^3 \cos \pi x \cos t] - \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[\psi_{xxx}]. \end{aligned} \quad (36)$$

Apply the inverse of natural transform to the equation (36).

$$\begin{aligned} \psi(x, t) &= \mathbb{N}^- \left[\frac{\sin \pi x}{s} \right] - \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[\sin \pi x \sin t + \pi^3 \cos \pi x \cos t] \right\} \\ &- \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[\psi_{xxx}] \right\}, \end{aligned} \quad (37)$$

$$\begin{aligned} \psi_0(x, t) &= \mathbb{N}^- \left[\frac{\sin \pi x}{s} \right] - \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[\sin \pi x \sin t + \pi^3 \cos \pi x \cos t] \right\} \quad (38) \\ &= \mathbb{N}^- \left[\frac{\sin \pi x}{s} \right] - \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+ \left[\sin \pi x \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots \right) + \pi^3 \cos \pi x \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots \right) \right] \right\} \quad (39) \end{aligned}$$

$$\psi_0(x, t) = \sin \pi x - \sin \pi x \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{t^{\alpha+5}}{\Gamma(\alpha+6)} - \dots \right) - \pi^3 \cos \pi x \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} \right). \quad (40)$$

$$\begin{aligned} \psi_{l+1}(x, t) &= -\mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[\psi_{lxxx}] \right\}. \quad (41) \\ l &= 0 \end{aligned}$$

$$\begin{aligned} \psi_1(x, t) &= -\mathbb{N}^- \left[\frac{u^\alpha}{s^\alpha} \mathbb{N}^+[\psi_{0xxx}] \right]. \quad (42) \end{aligned}$$

Now we compute each term:

$$\begin{aligned} \psi_{0xxx} &= \frac{\partial^3}{\partial x^3} (\psi_0(x, t)) = -\pi^3 \cos \pi x + \pi^3 \cos \pi x \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{t^{\alpha+5}}{\Gamma(\alpha+6)} - \dots \right) \\ &- \pi^6 \sin \pi x \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} \right) \\ \psi_1(x, t) &= -\mathbb{N}^- \left[\frac{u^\alpha}{s^\alpha} \mathbb{N}^+ \left[-\pi^3 \cos \pi x + \pi^3 \cos \pi x \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{t^{\alpha+5}}{\Gamma(\alpha+6)} - \dots \right) - \pi^6 \sin \pi x \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} \right) \right] \right] \\ \psi_1(x, t) &= \pi^3 \cos \pi x - \pi^3 \cos \pi x \left(\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} - \frac{t^{2\alpha+5}}{\Gamma(2\alpha+6)} - \dots \right) + \pi^6 \sin \pi x \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} - \frac{t^{2\alpha+4}}{\Gamma(2\alpha+5)} \right). \quad (43) \\ l &= 1 \end{aligned}$$

$$\psi_2(x, t) = -\mathbb{N}^- \left[\frac{u^\alpha}{s^\alpha} \mathbb{N}^+[\psi_{1xxx}] \right]. \quad (44)$$

Now we compute each term:

$$\begin{aligned} \psi_{1xxx} &= \frac{\partial^3}{\partial x^3} (\psi_1(x, t)) \\ &= \pi^6 \sin \pi x - \pi^6 \sin \pi x \left(\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} - \frac{t^{2\alpha+5}}{\Gamma(2\alpha+6)} - \dots \right) - \pi^9 \cos \pi x \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} - \frac{t^{2\alpha+4}}{\Gamma(2\alpha+5)} \right), \end{aligned} \quad (45)$$

$$\begin{aligned} \psi_2(x, t) &= -\mathbb{N}^- \left[\frac{u^\alpha}{s^\alpha} \mathbb{N}^+ \left[\pi^6 \sin \pi x - \pi^6 \sin \pi x \left(\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} - \frac{t^{2\alpha+5}}{\Gamma(2\alpha+6)} - \dots \right) - \pi^9 \cos \pi x \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} - \frac{t^{2\alpha+4}}{\Gamma(2\alpha+5)} \right) \right] \right], \end{aligned} \quad (46)$$

$$\begin{aligned} \psi_2(x, t) &= -\pi^6 \sin \pi x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \pi^6 \sin \pi x \left(\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - \frac{t^{3\alpha+3}}{\Gamma(3\alpha+4)} + \frac{t^{3\alpha+5}}{\Gamma(3\alpha+6)} + \dots \right) + \pi^9 \cos \pi x \left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)} + \frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)} + \dots \right), \end{aligned} \quad (47)$$

The NTDM solution for example 3 is

$$\begin{aligned} \psi(x, t) &= \psi_0(x, t) + \psi_1(x, t) + \psi_2(x, t) + \dots \\ &= \sum_{i=0}^n \psi_i(x, t), \end{aligned} \quad (48)$$

$$\begin{aligned} \psi(x, t) &= \sin \pi x - \sin \pi x \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{t^{\alpha+5}}{\Gamma(\alpha+6)} - \dots \right) - \pi^3 \cos \pi x \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} \right) \end{aligned}$$

$$\begin{aligned}
 & +\pi^3 \cos \pi x - \pi^3 \cos \pi x \left(\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} - \frac{t^{2\alpha+5}}{\Gamma(2\alpha+6)} - \right. \\
 & \left. \dots \right) + \pi^6 \sin \pi x \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} - \frac{t^{2\alpha+4}}{\Gamma(2\alpha+5)} \right) \\
 & - \pi^6 \sin \pi x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \pi^6 \sin \pi x \left(\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - \frac{t^{3\alpha+3}}{\Gamma(3\alpha+4)} + \right. \\
 & \left. \frac{t^{3\alpha+5}}{\Gamma(3\alpha+6)} + \dots \right) + \pi^9 \cos \pi x \left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)} + \right. \\
 & \left. \frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)} + \dots \right), \quad (49)
 \end{aligned}$$

For $\alpha = 1$ the exact form of $\psi(x, t)$ is

$$\psi(x, t) = \sin \pi x \cos t \quad (50)$$

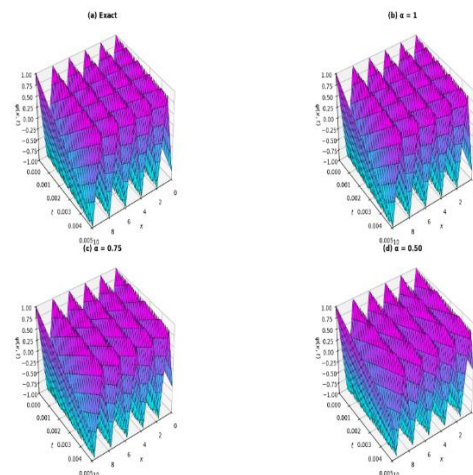


Figure 3: The (a) Exact and NTDM solutions of $\psi(x, t)$ of example 3 at (b) $\alpha = 1$, (c) $\alpha = 0.75$, and (d) $\alpha = 0.50$.

Example 4: consider the following non-homogeneous fractional dispersive KdV in equation in three dimensional spaces [18].

$$D_t^\alpha \psi(x, y, z, t) + \psi_{xxx} + \frac{1}{8} \psi_{yyy} + \frac{1}{27} \psi_{zzz} = \sin(x + 2y + 3z) \cos t - 3 \cos(x + 2y + 3z) \sin t, \quad 0 < \alpha \leq 1 \quad (51)$$

With the initial condition

$$\psi(x, y, z, 0) = 0. \quad (52)$$

Solution:

Apply the natural transform to the both side of equation.

$$\begin{aligned}
 & \mathbb{N}^+[D_t^\alpha \psi(x, y, z, t)] + \mathbb{N}^+[\psi_{xxx}] + \mathbb{N}^+\left[\frac{1}{8} \psi_{yyy}\right] + \\
 & \mathbb{N}^+\left[\frac{1}{27} \psi_{zzz}\right] = \mathbb{N}^+[\sin(x + 2y + 3z) \cos t] - \\
 & \mathbb{N}^+[3 \cos(x + 2y + 3z) \sin t] \\
 & \frac{s^\alpha}{u^\alpha} \mathbb{N}^+[\psi(x, y, z, t)] = \mathbb{N}^+[\sin(x + 2y + 3z) \cos t] - \\
 & \mathbb{N}^+[3 \cos(x + 2y + 3z) \sin t] - \mathbb{N}^+\left[\psi_{xxx} + \frac{1}{8} \psi_{yyy} + \right. \\
 & \left. \frac{1}{27} \psi_{zzz}\right] \\
 & \mathbb{N}^+[\psi(x, y, z, t)] = \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[\sin(x + 2y + 3z) \cos t] - \\
 & \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[3 \cos(x + 2y + 3z) \sin t] - \frac{u^\alpha}{s^\alpha} \mathbb{N}^+\left[\psi_{xxx} + \right. \\
 & \left. \frac{1}{8} \psi_{yyy} + \frac{1}{27} \psi_{zzz}\right]. \quad (53)
 \end{aligned}$$

Apply the inverse of natural transform to the equation (53).

$$\begin{aligned}
 \psi(x, y, z, t) &= \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[\sin(x + 2y + 3z) \cos t] \right\} - \\
 & \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[3 \cos(x + 2y + 3z) \sin t] \right\} - \\
 & \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+ \left[\psi_{xxx} + \frac{1}{8} \psi_{yyy} + \frac{1}{27} \psi_{zzz} \right] \right\} \\
 \psi_0(x, y, z, t) &= \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[\sin(x + 2y + 3z) \cos t] \right\} \\
 \psi_0(x, y, z, t) &= \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+ \left[\sin(x + 2y + 3z) \left(1 - \frac{t^2}{2!} \right. \right. \right. \\
 & \left. \left. \left. + \frac{t^4}{4!} + \dots \right) \right] \right\}
 \end{aligned}$$

$$\psi_0(x, y, z, t) = \sin(x + 2y + 3z) \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} + \dots \right), \quad (54)$$

$$\psi_{l+1}(x, y, z, t) = -\mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+[3 \cos(x + 2y + 3z) \sin t] \right\} - \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+ \left[\psi_{lxxx} + \frac{1}{8} \psi_{lyyy} + \frac{1}{27} \psi_{lzzz} \right] \right\}, \quad (55)$$

$$\psi_{l+1}(x, y, z, t) = -\mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+ \left[3 \cos(x + 2y + 3z) \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots \right) \right] \right\} - \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+ \left[\psi_{lxxx} + \frac{1}{8} \psi_{lyyy} + \frac{1}{27} \psi_{lzzz} \right] \right\}. \quad (56)$$

For $l = 0$

$$\psi_1(x, y, z, t) = -\mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+ \left[3 \cos(x + 2y + 3z) \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots \right) \right] \right\} - \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+ \left[\psi_{0xxx} + \frac{1}{8} \psi_{0yyy} + \frac{1}{27} \psi_{0zzz} \right] \right\}. \quad (57)$$

Now we compute each term:

$$\begin{aligned}
 \psi_{0xxx} &= \frac{\partial^3}{\partial x^3} (\psi_0(x, y, z, t)) = \frac{\partial^3}{\partial x^3} \left(\sin(x + 2y + 3z) \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} + \dots \right) \right) = -\cos(x + 2y + 3z) \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} + \dots \right) \\
 \psi_{0yyy} &= \frac{\partial^3}{\partial y^3} (\psi_0(x, y, z, t)) = \frac{\partial^3}{\partial y^3} \left(\sin(x + 2y + 3z) \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} + \dots \right) \right) = -8 \cos(x + 2y + 3z) \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} + \dots \right) \\
 \psi_{0zzz} &= \frac{\partial^3}{\partial z^3} (\psi_0(x, y, z, t)) = \frac{\partial^3}{\partial z^3} \left(\sin(x + 2y + 3z) \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} + \dots \right) \right) = -27 \cos(x + 2y + 3z) \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} + \dots \right), \quad (58)
 \end{aligned}$$

$$\begin{aligned}
 \psi_1(x, y, z, t) &= -\mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+ \left[3 \cos(x + 2y + 3z) \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots \right) \right] \right\} \\
 & - \mathbb{N}^- \left\{ \frac{u^\alpha}{s^\alpha} \mathbb{N}^+ \left[-\cos(x + 2y + 3z) \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} + \dots \right) + \frac{1}{8} \left(-8 \cos(x + 2y + 3z) \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} + \dots \right) \right) + \frac{1}{27} \left(-27 \cos(x + 2y + 3z) \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} + \dots \right) \right) \right] \right\}
 \end{aligned}$$

$$3z) \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} + \dots \right) \Bigg\}, \quad (59)$$

$$\begin{aligned} \psi_1(x, y, z, t) = & -N^- \left\{ \frac{u^\alpha}{s^\alpha} N^+ \left[3 \cos(x + 2y + 3z) \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots \right) \right] \right\} + N^- \left\{ \frac{u^\alpha}{s^\alpha} N^+ \left[\cos(x + 2y + \right. \right. \\ & 3z) \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} + \dots \right) - \left(\cos(x + 2y + \right. \\ & 3z) \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} + \dots \right) \Bigg\} - \left(\cos(x + 2y + \right. \\ & 3z) \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} + \dots \right) \Bigg\}. \quad (60) \end{aligned}$$

For $l = 0, 1, 2, 3, \dots$

$$\begin{aligned} \psi_1(x, y, z, t) &= 0, \\ \psi_{l+1}(x, y, z, t) &= 0. \end{aligned}$$

This already yields the exact solution and The NTDM solution for example 4 is

$$\begin{aligned} \psi(x, y, z, t) = & \sin(x + 2y + 3z) \left(\frac{t^\alpha}{\Gamma(\alpha + 1)} - \frac{t^{\alpha+2}}{\Gamma(\alpha + 3)} \right. \\ & \left. + \frac{t^{\alpha+4}}{\Gamma(\alpha + 5)} + \dots \right). \quad (61) \end{aligned}$$

For $\alpha = 1$ the exact form of $\psi(x, y, z, t)$ is

$$\psi(x, y, z, t) = \sin(x + 2y + 3z) \sin t, \quad (62)$$

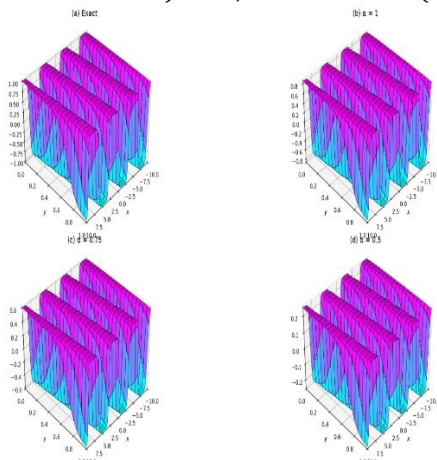


Figure 4: The (a) Exact and NTDM solutions of $\psi(x, y, z, t)$ of example 4 at (b) $\alpha = 1$, (c) $\alpha = 0.75$, and (d) $\alpha = 0.50$.

The NTDM solution coincides exactly with the analytical solution for $\alpha = 1$, demonstrating the accuracy and efficiency of the proposed method. Consequently, the absolute error between the exact and approximate solutions is zero up to machine precision, confirming the exact reproduction property of the method for this case.

V. CONCLUSION

In this paper, the analytical solutions of third order dispersive fractional partial differential equations are determined, using NTDM. The fractional derivatives are described by the Caputo operator. The NTDM,

solutions are obtained at fractional and integer orders for all problems. The results revealed the highest agreement with the exact solutions for the problems. The NTDM solutions for some numerical examples have shown the validity of the proposed method. It is also investigated that the fractional order solutions are convergent to the exact solution for the problems as fractional order approaches to integer order. The implementation of NTDM to illustrative examples have also confirmed that the fractional order mathematical model can be the best representation of any experimental data as compare to integer order model. Moreover, by taking different fractional orders, we can find a way to set suitable mathematical model for any experimental data, and thus found reasonable consequences. Hence, it is concluded that NTDM is the best tool for the solution of FPDEs.

5.1. Conflicts of Interest: The authors have no conflict of interest.

5.2. Authors contribution:

- Noorullah Noori conceptualized the study, developed the methodology, supervised the research, and reviewed the manuscript.
- Naqibullah Darwish carried out the mathematical analysis, implemented the Natural Transform Decomposition Method, performed the numerical computations, and prepared the initial draft of the manuscript.
- Noorullah Noori, Naqibullah Darwish, and Abdul Wakil Baidar jointly analyzed and interpreted the results, verified the mathematical derivations, and critically revised and approved the final manuscript.

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